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Title

Thermal analysis of rooms with diurnal periodic heat gain, ThermSim
Part 1 : Derivation

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Synopsis

Temperature and cooling demand in a room summertime is influenced by numerous factors like : internal gains, ventilation, solar gain, behaviour of occupants, thermal inertia of the room, and outdoor conditions (climate).

The thermal environment and cooling demand summertime is often analysed using advanced computer programs. These programs require detailed input describing every feature of the room. Often the overview, transparency and some of the physical insight is lost using these advanced computer programs.

In a predesign phase of a project it is preferable to do simple calculations of the thermal behaviour of a room. These simple calculations often gives more physical insight and overview than using computer programs. Simple calculations also gives a quality assurance of later computer analysis of the room.

In this paper a simplified thermal analysis of a room is presented, called *ThermSim*, which can be used as a hand calculation method in the predesign phase of a project.

In rooms with significant solar gain, the total heat gain to the room at any time of day, can be approximated with a simple cosine function. This assumption together with a thermal one-mass-model, and a frequency analysis model often used in electric circuits analysis, forms the basis of the thermal room model. The solution of this model gives a simple equation which can predict the temperature in the room, exposed for a heat wave midsummer.

The method shows in a transparent way the time-lag between maximum heat gain and maximum occurring room temperature. I addition the "thermal build up" in a heavy room from day to day during a heat wave is easily predicted. ThermSim is compared (comparison found in part 2) with more advanced computer analysis and shows good agreement when the model assumptions is fulfilled.

List of symbols

Symbol	Description	Unit
A_{fac}	Facade area	m^2
A_{win}	Area for whole window construction (including frame)	m^2
C_{air}	Heat capacity of air (can be set to 0.34 Wh/kgK)	Wh/kgK
L	Mechanical or natural air flow rate	m^3/h
n	Air infiltration in ACH	1/h
$\bar{q}$	Daily mean heat gain	W
$\hat{q}$	Daily amplitude heat gain	W
t	Time	h

t_{max}	Time for maximum heat gain and external temperature to occur	h
T_i	"Effective" room temperature	°C
$\bar{T}_e$	Mean daily external temperature	°C
$\hat{T}_e$	Daily amplitude external temperature	°C
U_{fac}	U-value facade construction	W/m²K
U_{win}	U-value window construction	W/m²K
V	Room air volume	

1 Introduction

Thermal design of rooms are often done using advanced computer simulation tools. These tools are often cumbersome to use and give little insight in the physical process which is simulated. In the early stage of a design process it can be beneficial to use simple hand calculation for a rough predesign of a room. This gives a much better physical insight to the thermal process in the room. In addition it can be a valuable quality assurance for later computer simulations.

This paper describes a simplified method for simulation of temperatures and cooling load, called *ThermSim*, which can be used for hand calculation or it can easily be implemented in a spreadsheet. This method can be used on most rooms provided they have a daily variation in the heat gain that can be approximated with a sinusoidal variation in the gain. This is often the case with rooms exposed to solar radiation.

The method shows in a transparent way how the temperature evolves from day to day during a heat wave. In addition it gives the daily variation in room temperatur, with maximum and minimum room temperature. Or, it can be used to estimate necessary cooling load for keeping the temperature and daily temperature variation at an acceptable level.

This paper, part 1, derives the method and interpret the different terms in the model. Part 2, which is given in a accompanying paper, contains tables which simplifies the use of the method, along with examples and comparison to advanced computer simulations.

The model is based on a five main assumptions :

1. Daily variation in heat gain and external temperature is approximated by a sinusoidal function
2. Room air temperature, surface temperatures and "building structure" temperature is "merged" into one mean effective room temperature
3. The effective heat capacity of the room is limited to a finite thickness of the building constructions
4. Every input to the model is either constant or approximated with a diurnal sinusoidal variation
5. Heat loss to adjacent rooms are negligible

2 Formulation of model

2.1 Physical and mathematical derivation

In summertime rooms with a external windows are exposed to a diurnal variation in heat gain. This variation can often, with good approximation, be estimated with a sinusoidal function on the form :

$$q(t) = \bar{q} + \hat{q}\cos\left(\frac{2\pi(t - t_{\max})}{24}\right) \qquad \text{(W)} \qquad (1)$$

q(t) is heat gain in Watt as a function of time (t), $\bar{q}$ is the daily mean heat gain (W), $\hat{q}$ is the heat gain amplitude[1] (W), t is time (hours) and t_{max} is the time when maximum heat gain occur. The period is of course 24 hours.

If there is a ventilation system (natural or mechanical) supplying the room with the air flow rate L_{vent} the cooling effect (heat loss) is :

$$q_{vent} = C_{air}L_{vent}\left(T_i - T_e\right) = H_{vent}\left(T_i - T_e\right) \qquad \text{(W)} \qquad (2)$$

C_{air} is the volumetric heat capacity of air which can be set to 0.34 Wh/m³K, L is the air flow in m³/h , T_i is the room air temperature (°C) and T_e is the external temperature (see below). Air flow is assumed constant, and mechanical cooling (cooling coil) is not considered here (treated separately in section 2.3). Heat gain from fans in mechanical ventilation has to be added to the other heat gains in equation (1).

Heat loss to the external that can be written :

$$q_{ext} = \left[C_{air}nV + \sum U_{win}A_{win} + \sum U_{fac}A_{fac}\right]\left(T_i - T_e\right) = H_{ext}\left(T_i - T_e\right) \quad \text{(W)} \quad (3)$$

where U_{win} is the U-value of the window and A_{win} is the window area including the frame(m²), n is the infiltration rate in ACH and V is the room air volume (m³), U_{fac} is the U-value for the facade construction and A_{fac} is the facade area.

The external temperature (T_e) in equation (2) and (3) , varies during the day, and this variation can be estimated with a sinusoidal function in the same form as (1) :

$$T_e(t) = \overline{T}_e + \hat{T}_e\cos\left(\frac{2\pi(t - t_{\max})}{24}\right) \qquad \text{(°C)} \qquad (4)$$

[1] The amplitude can be taken as the difference between the maximum heat gain q_{max} and the minimum heat gain q_{min} divided by two :

$$\hat{q} = \frac{q_{\max} - q_{\min}}{2}$$

If the room temperature fluctuates there will be heat accumulation in the building structure, and to some extent in the room air. If the roomtemperature rises dT_i during a small timespan dt, the heat accumulation is :

$$q_{acc} = \left(\sum A_{sur} C_a'' + C_{air} V\right) \frac{dT_i}{dt} = C_a \frac{dT_i}{dt} \qquad \text{(W)} \qquad (5)$$

Where A_{sur} is the area of all surfaces in the room having significant heat capacity, C_a" is the effective heat capacity pr. square meter for the surface (the specific heat capacity of the accumulating layer in the construction), dT_i is the infinitesimal temperature rise during the infinitesimal timestep dt.

We are now ready to formulate the heat balance for the room. According to the first law of thermodynamics heat gain minus heat loss will equal heat accumulation :

$$\overbrace{q_{gain}(t)}^{Gain} - \overbrace{\left(q_{vent} + q_{ext}\right)}^{Losses} = \overbrace{q_{acc}}^{Accumulation}$$

or

$$\overbrace{\overline{q}_{gain} + \hat{q}_{gain} \cos\left[\omega\left(t - t_{\max}\right)\right]}^{Gain} - \overbrace{\left(H_{ext} + H_{vent}\right)\left(T_i - \overline{T}_e - \hat{T}_e \cos\left[\omega\left(t - t_{\max}\right)\right]\right)}^{Losses} = \overbrace{C_a \frac{dT_i}{dt}}^{Accumulation}$$

For mathematical convenience it can be more compactly written as :

$$\frac{dT_i}{dt} + \frac{T_i}{\tau} = \frac{T_\infty}{\tau} + \delta \cos\left[\omega\left(t - t_{\max}\right)\right] \qquad \text{(K/h)} \qquad (6)$$

where the new parameters : the frequency ω, the timeconstant τ, the stationary temperature T_∞ and the amplitude coefficient δ have been introduced, which is given by :

$$\omega = \frac{2\pi}{24} \qquad \text{(1/h)} \qquad (7a)$$

$$\tau = \frac{C_a}{H_{vent} + H_{ext}} \qquad \text{(h)} \qquad (7b)$$

$$T_\infty = \frac{\left(H_{ext} + H_{vent}\right)\overline{T}_e + \overline{q}}{H_{vent} + H_{ext}} \qquad (°\text{C}) \qquad (7c)$$

$$\delta = \frac{\left(H_{ext} + H_{vent}\right)\hat{T}_e + \hat{q}}{C_a} \qquad \text{(K/h)} \qquad (7d)$$

2.2 Solution

The solution to equation (6) can be written in the closed form :

$$T_i(t) = \overbrace{\Delta T e^{-t/\tau}}^{Transient} + \overbrace{\overline{T}_\infty}^{Stationary} + \overbrace{\hat{T}\cos\left[\omega\left(t - \tau_{lag} - t_{max}\right)\right]}^{Periodic} \qquad (°C) \qquad (8)$$

For a more detailed mathematical derivation of (8) see the appendix. The solution is the superposition of a transient temperature, a mean stationary temperature, and a diurnal periodic temperature. T_∞ is the mean stationary value defined above, ΔTa is the transient temperature-difference given by :

$$\Delta T = T(0) - T_\infty - \hat{T}\cos\left[\omega\left(t_{max} - \tau_{lag}\right)\right] \qquad (K) \qquad (9)$$

T(0) is the intial temperature (00.00 the first day). The temperature amplitude $\hat{T}$ related to the .periodic temperature variation is given by:

$$\hat{T} = \frac{\delta}{\sqrt{\tau^{-2} + \omega^2}} \qquad (K) \qquad (10)$$

For interpretation of the solution it is wise to introduce the parameter time-lag, defined by :

$$\tau_{lag} = \frac{\arctan(\tau\omega)}{\omega} \qquad (h) \qquad (11)$$

2.3 Estimation of cooling load

If the calculated temperature is exceeding accepted limits, we need to estimate necessary cooling capacity to maintain comfortable temperature conditions in the room. By cooling capacity we mean mechanical cooling, which comes in addition to cooling by the external air flow rate (mechanical, natural or infiltration).
Equation (8) can be used to calculate the cooling load if we do the following simplification :

- We assume that diurnal periodic stationary condition has been reached. It implicates that the transient term in equation (8) has become negligible.

The total cooling load can then be estimated by the sum of the to effects :

1. Of removing so much of the mean heat gain that stationary mean temperature is reduced to a desired mean temperature $\overline{T}_{cool}$. This load can be called the mean cooling load, denoted : $\overline{q}_{cool}$

2. And removing so much of the amplitude heat gain that the temperature amplitude is reduced to a desired temperature amplitude $\hat{T}_{cool}$. This cooling load can be called the amplitude cooling load, denoted : $\hat{q}_{cool}$

The mean cooling load can be found by setting the stationary term in (8) equal to the desired mean cooling temperature $\overline{T}_{cool}$, and reducing the heat gain with $\overline{q}_{cool}$:

$$\overline{q}_{cool} = \left(H_{ext} + H_{vent}\right)\overline{T}_e + \overline{q} - \left(H_{ext} + H_{vent}\right)\overline{T}_{cool} \qquad \text{(W)} \qquad (12)$$

To reduce the amplitude temperature variation around the stationary mean temperature to a desired level (e.g. 2 °C), we have to remove the heat :

$$\hat{q}_{cool} = \left(H_{ext} + H_{vent}\right)\hat{T}_e + \hat{q} - \hat{T}_{cool} C_a \sqrt{\tau^{-2} + \omega^2} \qquad \text{(W)} \qquad (13)$$

where $\hat{T}_{cool}$ is the allowed temperature amplitude (variation). The maximum total cooling load is then given as :

$$q_{cool} = \overline{q}_{cool} + \hat{q}_{cool} \qquad \text{(W)} \qquad (14)$$

3 Discussion and interpretation

As seen in equation (8) there are three terms in the solution, that are significant to the resulting temperature. We have three constants with the unit of temperature (°C) : the transient temperature difference ΔT, the stationary temperature $\overline{T}$ and the temperature amplitude $\hat{T}$. In addition there are two constants with unit time (hours), the timeconstant τ and the time-lag τ_{lag}. In the following these constants, and how they influence the solution, will be discussed.

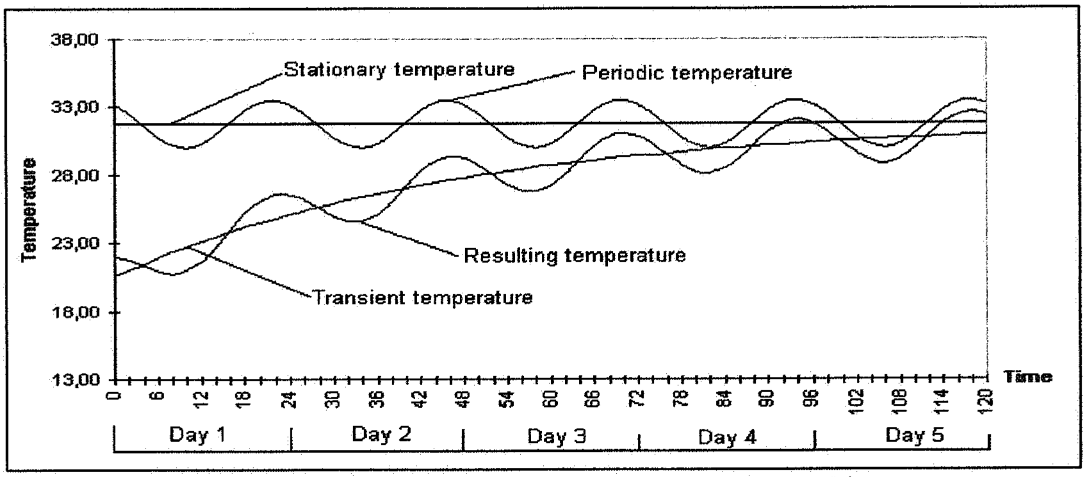

Figure 1 : Graphic illustration of the three terms in eq. (8), and the resulting roomtemperature

3.1 Transient term
The transient term given by :

$$\overbrace{\Delta T e^{-t/\tau}}^{Transient} \qquad (°C) \qquad (12)$$

determine how fast periodic stationary temperatures in the room is reached. ΔTa is the difference between the initial temperature (00.00) before the heatwave began, and the maximal occurring temperature at time 00.00 (after a long time). Since the initial temperature is lower than the stationary temperature, ΔTa will always be negative!
Typical values of ΔTa is between -5 °C and -20 °C.

How fast the transient term vanishes, that is how fast stationary conditions are reached, is entirely determined by the timeconstant τ. The timeconstant can vary from a few hours to more than 100 hours, depending on how large the thermal inertia of the room is. Table 1 shows how many percent of the temperature difference that is left, after 1, 2, 3 and 4 timeconstants have elapsed. After 3 timeconstants have elapsed, 95 % of the temperature difference is vanished, which can be a practical limit for the stationary level. For example : if the temperature difference is ΔTa = - 10°C and the timeconstant is 20 hours, the temperature would have rised 9.5 °C of possible 10°C after 60 hours.

Table 1 : Shows how much of the temperature difference (ΔTa) is left after 1, 2, 3 and 4 timeconstants (τ)

Elapsed time after number of timeconstants	1	2	3	4
% of temperature difference left	36.8	13.5	5.0	1.8

3.2 Stationary term
The stationary term is given by :

$$T_\infty = \frac{(H_{ext} + H_{vent})\overline{T}_e + \overline{q}}{H_{ext} + H_{vent}} \qquad (°C) \qquad (13)$$

As can be seen from (27) the stationary temperature is independent of the thermal capacity of the room. Furthermore we see that large mean heat gain ($\overline{q}$), and high mean external temperature ($\overline{T}_e$) leads to high stationary temperature (T_∞).

3.3 Periodic term
The periodic term :

$$\hat{T}\cos\left[\omega\left(t - \tau_{lag} - t_{\max}\right)\right] \qquad (°C) \qquad (14)$$

gives raise to periodic temperature oscillation. Equation (28) gives periodic oscillation with temperature amplitude $\hat{T}$, and a time-lag (τ_{lag}) . That is, the maximum temperature occur the time τ_{lag} after the time t_{max} , which is the time for maximum heat gain (and external temperature). Both $\hat{T}$ and τ_{lag} depends strongly on the thermal inertia of the room. Rooms with large thermal inertia gives small temperature amplitudes (as expected), and an increase in the time-lag. $\hat{T}$ is also depending on the heat gain amplitude and external temperature amplitude.

3.4 Maximum possible room temperature

The maximum possible temperature in the room occur when diurnal stationary condition is reached, together with daily maximum temperature amplitude :

$$T_{\max} = \overline{T}_{\infty} + \hat{T} \qquad (°C) \qquad (15)$$

This maximum possible temperaterature is likely to occur in rooms with small thermal inertia (small heat capacity and large heat loss). For rooms with large thermal inertia (large heat capacity and small heat loss) stationary condition is seldom reached before the heat wave is over (or has been reduced). In offices or other rooms with a weekly 5 day occupation, the simulation period is often set to 5 days. In these rooms the maximum possible temperature (T_{max}) is unlikely to occur (also see the accompanying paper, part 2).

4.0 Conclusion

- We have presented a model which simulates temperature and cooling loads in rooms with diurnal variation of the heat gain
- The model is an alternative to use of advanced simulation tools, in an early stage of the thermal design phase of a room
- In rooms with small thermal inertia (small heat capacity and large heat loss) diurnal stationary condition ia reached fast (2 - 4 days) (see part 2)
- In rooms with large thermal inertia (large heat capasity and small heat loss) diurnal stationary conditions are seldom reached during a normal heat wave(5- 10 days), or during a normal 5 days working week (see part 2)
- Large heat capasity reduce the daily temperature variation to a large extent, and reduce the cooling demand (caused by high peaks in the heat gain e.g. solar gain)
- This implicate that use of heavy building structure can reduce temperature problems and mechanical cooling demand summertime, provided that the temperature is allowed to fluctuate

References

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Appendix : Mathematical derivation

Equation (6) is a first order nonhomogeneous differential equation which can be solved with a variety of methods, such as ; Undetermined coeffisients Laplace transform, D-operator method, integrating factor method and substitution methods.

In all cases the general solution to (6) is the sum of the complementary solution, related to the left hand side (homogeneous part) , and the particular solution related to the right hand side (nonhomogeneous part).

Complementary solution
The left hand side in equation (6), which is called the homogeneous part, can easily be separated and integrated :

$$\frac{dT_{i,C}}{dt}+\frac{T_{,Ci}}{\tau}=0 \Leftrightarrow \frac{dT_{i,C}}{T_{i,C}}=-\frac{dt}{\tau} \Leftrightarrow T_{i,C}(t)=C_1 e^{-\frac{t}{\tau}} \quad , \; C_1 = \text{constant} \tag{A.1}$$

Particular solution
The particular solution can be found by using the method of undetermined coefficients. We then assume a particular solution in the same algebraic form as the right hand side of equation (6) :

$$T_{i,P} = A + B\cos\omega\left(t - t_{\max}\right) + C\sin\omega\left(t - t_{\max}\right) \tag{A.2}$$

Differentiating (A.2) and substituting into (6) and collecting coefficients, we find A, B and C to be :

$$A = \beta\tau = T_\infty \quad ; \quad B = \frac{\alpha\delta}{\alpha^2+\omega^2} \quad ; \quad C = \frac{\omega\delta}{\alpha^2+\omega^2} \tag{A.3}$$

General solution
The general solution is then given by the sum of the complementary and the particular solution :

$$T_i(t) = C_1 e^{-\frac{t}{\tau}} + T_\infty + \frac{\delta}{\tau^{-2}+\omega^2}\left(\omega \sin\left[\omega\left(t-t_{\max}\right)\right] + \frac{1}{\tau}\cos\left[\omega\left(t-t_{\max}\right)\right]\right) \tag{A.4}$$

(A.4) can be written more conveniently by a trigonometric identity[2] as :

$$T_i(t) = C_1 e^{-\alpha t} + T_\infty + \hat{T}\cos\left[\omega\left(t-\tau_{lag}-t_{\max}\right)\right]$$

$$\tau_{lag} = \frac{\arctan(\tau\omega)}{\omega} \quad ; \quad \hat{T} = \frac{\delta}{\sqrt{\tau^{-2}+\omega^2}} \tag{A.5}$$

The constant C_1 can be determined by the initial condition $T_i(t=0) = T(0)$:

$$C_1 = \Delta T_a = T_a(0) - \overline{T}_{a,\infty} - \hat{T}_a \cos\left[\omega\left(t_{\max}-\tau_{lag}\right)\right] \tag{A.6}$$

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[2] We have that : $A\sin w + B\cos w = \sqrt{A^2+B^2}\cos(w-\phi) \quad ; \quad \tan\phi = \frac{A}{B}$