VENTILATION STRATEGIES AND MEASUREMENT TECHNIQUES

6th AIC Conference, September 16-19 1985, Netherlands

PAPER S.3

INTERPRETATION AND ERROR ANALYSIS OF MULTI-TRACER GAS MEASUREMENTS TO DETERMINE AIR MOVEMENT IN A HOUSE

R R Walker

Building Research Establishment
Garston
Watford, WD2 7JR
United Kingdom

INTERPRETATION AND ERROR ANALYSIS OF MULTI-TRACER GAS MEASUREMENTS TO DETERMINE AIR MOVEMENT IN A HOUSE.

by R R Walker

1. INTRODUCTION

Although infiltration of outside air across the envelope of a building has been considered of prime interest in relation to energy conservation and indoor air quality, it also important to understand the way in which air moves between zones within a building. A knowledge of the air movement pattern enables the transfer of pollutants or heat to be determined. In order to achieve this, a number of experimental methods have recently been developed, using either single or multiple tracer gases. (See, for instance, references 1,2,6,7,9).

It is important in any method to assess the confidence which can be placed in the resulting flow rates. This paper discusses methods for analysing and evaluating errors arising from measurements made using three tracer gases. The test data presented are taken from a programme of measurements to determine the infiltration rates and air interchange between three zones of a mehanically ventilated experimental house. The full programme is designed to investigate the effect of operating the mechanical ventilation system and the use of internal doors, as well as meteorological factors.

2. THE TEST HOUSE

The house was built on site at Garston in 1978 as a 'low energy' test house, and incorporates a high standard of insulation and a mechanical ventilation system with an air to air heat exchanger for heat recovery. The elevations and floor plans are shown in Figures 1 and 2.

The mechanical ventilation system supplies fresh air via ducting to the three bedrooms and the living room. Stale air is similarly extracted from the bathroom, downstairs toilet, kitchen dining area and from the cooker hood.

The supply and extract flow rates have been measured to be approximately 81m^3/hr to the bedrooms and 41m^3/hr to the lounge. The individual extract rates have not been measured, and are esimated to be approximately in the ratio 1:3 from the bathroom upstairs and from both kitchen and toilet downstairs. The total supply and extract rates are in balance.

3. INJECTION STRATEGY

For the present purposes the house was notionally divided into three zones; the ground floor, first floor and loft space, called zones 1, 2 and 3 respectively. Zones 1 and 2 have a nominal volumes of 97.5m^3. The loft space, which is common to all three houses and extends across the whole terrace, has a volume of 260m^3.

Three tracer gases, CO_2, N_2O and SF_6 were used, one initially distributed throughout each zone. CO_2 was injected to a target of 2000-5000ppm in the ground floor, N_2O to 200ppm in the first floor, and SF_6 to either 50 or 200ppm in the loft. These choices were determined by the available analysers, described below.

Tracer gas was delivered to each zone via single 4mm ID nylon tubes. In zone 1 this then branched into each room and the hall. There were similar networks for zones 2 and 3.

The tracer gas delivered by each branch was dispersed through fans, and the delivery rate adjusted with needle valves. Fans were used during injection to aid mixing of tracer gas within each zone. These consisted of pairs blowing in opposite directions, located on the thresholds of each room and along the loft space. In addition, oscillating desk top fans were placed in the centre of each room.

Injection times for each tracer were established by trial and error. These were all less than one minute. The mixing fans were switched off fifteen minutes after injection.

4. SAMPLING OF TRACER GASES

The analysers used were two continous output, dual channel, non-dispersive infrared instruments by Leybold Heraeus. Channels were dedicated to analysis of either one of SF_6, N_2O or CO_2. Depending on the unit used, SF_6 analysis was on either a 200ppm or a 50ppm range, and injection times were altered accordingly.

Air samples were drawn from each zone via a network of tubing, which exactly mirrored the injection network. Thus samples from each location within a zone were blended before passing back to the analysers. In addition, a single line ran to outside to obtain a reference level for each trace analysis. Two further 'flying leads' were used to obtain individual ground and first floor room samples. In this way the evenness of initial concentration levels within each zone could be checked, and the tracer injection rate set accordingly.

The total of six sample lines were then brought back to individual solenoid valves, which were under the control of an ITT Director microprocessor unit. This unit was programmed to connect each sample line in turn to the two analysers. The concentrations of the three tracers present in each sample were recorded on cassettes by a data logger unit, using an arbitrary scale of 0 - 200 units. The data were later transcribed using an off-line computer. A shematic layout of this system is shown in Figure 3.

Tests began by switching off the mixing fans and by starting the sampling system. Each test continued for thirty minutes.

5. THEORY

The theoretical basis for deriving ventilation and interzone airflow rates from measurements of multiple tracers is detailed in Reference 1. It was shown how in the 'decay method' the conservation of tracer gas in a zone (k) can be written in the form:

$$[A].\{X\}_{(k)} = \{B\}_{(k)} \qquad .....(1)$$

where the corresponding elements are, respectively:

$$a_{ij} = C_{ij}$$

$$b_{i(k)} = V_{(k)} \cdot \dot{C}_{i(k)}$$

$$x_{i(k)} = -S_{(k)} \cdot \delta(ik) + Q_{i(k)} \cdot (1 - \delta(ik))$$

where $\delta(ik)$ is the Kronecker delta, and the order of the matrices is equal to the number of zones, n. The measurable quantities are:

$V_{(k)}$ = the volume of zone (k)

C_{ij} = the concentration of tracer (i) in zone (j)

$\dot{C}_{i(k)}$ = the time derivatives of the concentrations $C_{i(k)}$

which are used to solve for the unknowns:

$S_{(k)}$ = the total outflow of air from zone (k)

Q_{ij} = the flow from zone (i) to zone (j)

and subsequently also for the infiltration (Q_{0i}) and exfiltration (Q_{i0}) terms.

Equation (1) can be replaced by n(n+1) simultaneous equations describing the mass balance of n tracers and air, in terms of n(n+1) unknown exchanges between n zones and the outside. This is considered later in Section 11, and Appendix B.

It will be noted that in order to solve the equations in their present form, the input data require the 'slopes' of the $C_{ij}(t)$ curves to be established for each tracer in each zone. This will be referred to as the Gradient Method. An alternative approach is to integrate the mass balance equations throughout with respect to time, over some period τ. τ might be chosen to be the whole duration of the test for example. The flow matrix remains unaffected, but the elements of [A] and {B} become:

$$a_{ij} = \int_{\tau} C_{ij} \, dt$$

$$b_{i(k)} = V_{(K)} \cdot \Delta C_{i(k)}$$

where

$\int_{\tau} C_{ij}.dt$ = the area under the curve $C_{ij}(t)$ of the tracer(i) recorded in zone (j), over the period τ.

$\Delta C_{i(k)}$ = change in concentration of tracer (i) in zone (k) over the duration τ.

There will be a 'smoothing effect' on the data by using the equations in this integrated form. This will be referred to as the Integral Method.

6. DATA ANALYSIS

For the Gradient Method the concentrations and derivatives at a single time point were obtained from the concentration profiles plotted on a semilogarithmic scale. A straight line was drawn through data points local to the specified time. The derivative was then computed from the concentration at that time, on the line fit, and the slope of the line.

For the Integral Method the curves were integrated numerically over a specified period using a simple trapezoidal method. The overall changes in concentration over this period were also noted.

The data, in either time derivative or time integral form, were then used to solve for the flows by computer, using the Gauss elimination method. It is possible for small negative values to be computed; these have no physical interpretation in the present context. A least squares procedure[2] advocated by Penman and Rashid[3] was available to constrain the solutions to have positive values. For the purposes of error analysis, this was not used.

7. ERROR ANALYSIS

The aims of the data analyses in the following Sections are fivefold:

1. to discuss processsing by the Gradient Method and Integral Method

2. to evaluate three schemes of error analysis, based on vector norms, perturbation, and differentiation, respectively

3. to discuss the role of reconstructed concentration profiles in validating solutions

4. to compare the performance of the schemes of error analysis against other published data.

5. to establish confidence levels in the flow solutions

All error estimates are based on the fluctuations in the concentration profiles, which represent zonal averages in the sense that they are measurements of physically blended air samples. These fluctuations were taken to approximately average +/-1 logger units. In the time integration procedure, say over N points, the error was assumed to sum in proportion to $1/\sqrt{N}$, to take some account of the 'smoothing' effect. Over this period the net change in concentration of tracer was estimated to be accurate within +/-2 units in the zone of seeding, and +/-1 unit elsewhere. Errors in derivatives using the Gradient Method were estimated visually.

8. AIRFLOW RESULTS

Figure 4 shows the concentration profiles of CO_2 recorded in Test 1. The reconstructed curves are also shown and are discussed in Section 13. The data extracted for processing by the Integral Method are given in Table 1, and for processing by the Gradient Method in Table 2.

For Tests 1 and 2, solutions were obtained using the Integral Method over thirty minutes (I_{30} data) and fifteen minutes (I_{15} data), and the Gradient Method at eleven minutes elapsed time (G_{11} data) and at five minutes (G_5 data). These are listed in Table 3 for Test 1, and in Table 4 for Test 2. The mean and standard deviations were calculated for these solutions. For Test 1 these are listed in Table 5. The G_5 solutions were discounted for reasons discussed below.

Where a particular concentration profile departs significantly from a smooth curve, then it is very difficult to measure an appropriate derivative. In Test 1 a low estimate of the decay rate of N_2O in the loft, at five minutes, led to values being computed for Q_{03} and Q_{30} which were approximately 50% lower than the mean results. Similar discrepencies are apparent in the G_{11} results of Test 2.

The problem arises because a curve was fitted over a limited number of data points only. Other researchers (I'Anson, Irwin and Howarth[6], and Prior et al[7]) have used procedures which entail a theoretical curve being fitted to the whole of the data. However, it is not always possible to fit an appropriate curve, and these procedures are not without problems. It is suggested that repeated solution at several time points would be an improvement on the method presented here, perhaps using finite difference techniques to obtain the gradients.

Solutions obtained using integration are not subject to such large variations. In Tests 1 and 2, the solutions obtained by integration exhibited less variation about the mean.

9. THEORY OF ERROR ANALYSIS USING VECTOR AND MATRIX NORMS.

A procedure for the rigorous error analysis of matrix processes is given by Wilkinson[4,5]. The following is a brief summary of this procedure, which is presented more fully in Appendix A.

Use is made of vector and matrix norms. The norm gives an assessment of the size of a vector or matrix. The (infinity') norm, $\|X\|$, of a vector $\{X\}$ is interpreted as the modulus of the largest element. Corresponding to this vector norm, the matrix norm A is defined as the maximum row sum of the moduli of the elements.

With reference to equation 1, perturbations in the matrix [A] and the right-hand sides 'B' are considered. The following expression is derived:

$$\|\delta X\| \leqslant \frac{\|A^{-1}\|.\|\delta A\|.\|X\| + \|A^{-1}\|.\|\delta B\|}{1 - \|A^{-1}\|.\|\delta A\|} \quad \ldots.(2)$$

This result provides an upper bound for the largest expected perturbation in the elements of $\{X\}_{(k)}$, due to perturbations $[\delta A]$ and $\{\delta B\}_{(k)}$ in the elements of $[A]$ and $\{B\}_{(k)}$, respectively.

It is instructive to consider equation 2 expressed in terms of relative errors:

$$\frac{\|\delta X\|}{\|X\|} \leqslant \frac{\|A\|.\|A^{-1}\|.\left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta B\|}{\|B\|}\right]}{1 - \|A\|.\|A^{-1}\|.\frac{\|\delta A\|}{\|A\|}} \qquad \ldots.(3)$$

It can be seen that a decisive quantity is $\|A\|.\|A^{-1}\|$. This expresses the sensitivity of the solution to perturbations in the parameters, and is termed a 'condition number' for the problem. Ill-conditioning is indicated by $\|A\|.\|A^{-1}\|$ much greater than unity.

Expression 2 was evaluated for the results of four tests. As an example, results are listed for the Test 1, I_{30} data in Table 5. In any zone, $\{\delta X\}$ appears pessimistic compared to the standard deviations of the flow solutions, especially in zone three. Error norm limits computed from the data of the Gradient Method were generally larger.

For the Integral Method the greater proportion of the computed error bound was due to the uncertainty in the measurement of the net changes in concentration. For the Gradient Method, the dominant contribution came from the uncertainties in the derivatives.

The values computed cannot strictly be ascribed to, or distributed amongst, any particular airflows. At best they are an indication of the possible size of errors in the flows.

10. ERROR ESTIMATION USING PERTURBATION OF DATA.

The sensitivity of the solutions to changes (perturbations) in the data can be checked directly by making small changes in the data, and then computing new (perturbed) solutions. A particular set of perturbations could be chosen, say all errors occuring together with the same sign. This can be taken a step further, to take into account all possible combinations of errors, with the proposed scheme as follows.

Consider any airflow solution Q_{ij} as a function of all measured quantities a_{ij}, $b_{i(k)}$

$$Q_{ij} = f(a_{ij}, b_{i(k)})$$

A small change (δQ_{ij}) in Q_{ij}, due to small changes (δa_{ij} and $\delta b_{i(k)}$) in a_{ij} and $b_{i(k)}$ can be expressed using partial differentials:

$$\delta Q_{ij} \approx \sum_{i,j} \left[\frac{\partial f}{\partial a_{ij}} \cdot \delta a_{ij} + \frac{\partial f}{\partial b_{i(k)}} \cdot \delta b_{i(k)} \right]$$

where f is assumed to be approximately linear over the small changes δa_{ij} and $\delta b_{i(k)}$. Approximating the differentials themselves by the small changes δf_{ij} and δa_{ij}, and $\delta f_{i(k)}$ and $\delta b_{i(k)}$, we obtain:

$$\delta Q_{ij} \approx \sum_{ij} (\delta f_{ij} + \delta f_{i(k)})$$

$$|\delta Q_{ij}| \leqslant \sum_{ij} (|\delta f_{ij}| + |\delta f_{i(k)}|) \qquad \ldots.(4)$$

This says that the size of the error, $+/-\delta Q_{ij}$, in a solution Q_{ij} is less than the sum (without regard to sign) of all the small perturbations in Q_{ij}, found by making small changes in all a_{ij} and $b_{i(k)}$ in turn.

The results of this scheme for the I_{30} of Test 1 are presented in Table 5. For the largest airflow, $188 m^3/hr$, the computed error was $72 m^3/hr$. 92% of this error is due to errors in $\{\delta B\}$, i.e. in the estimation of the nett changes in concentration.

11. ERROR ANALYSIS BY DIFFERENTIATION - MATRIX FORM

It was stated above that the basic equations can be written in the form of n(n+1) simultaneous equations, so as to explicitly include exchanges between the zones and the outside air mass:

$$\{B\} = [A].\{X\}$$

It is difficult to derive a systematic notation which defines each of the matrix and vector elements. It was found necessary to set out the equations in full, as given in Appendix B.

Using a standard technique of error analysis and differentiating (as outlined by Perera[10]) we obtain, in matrix notation:

$$\{dB\} = [A].\{dX\} + [dA].\{X\}$$

$$\{dX\} = [A^{-1}].(\{dB\} - [dA].\{X\}) \qquad \text{.............}(5)$$

The appropriate absolute errors are inserted in place of the infinitessimal quantities $\{dB\}$ and $[dA]$, and $[A^{-1}]$ is computed from $[A]$. Matrix computations are then made, without regard to signs, to compute error limits $\{dX\}$.

This procedure was performed for the I_{30} data of Test 1, and the results are listed in Table 5. These do not significantly differ from the results obtained using the perturbation scheme above.

12. PERFORMANCE OF SCHEMES OF ERROR ANALYSIS USING OTHER PUBLISHED DATA.

The author is unaware of any published results of measurements of interzonal air movement which include a full error analysis of results. However D'Ottavio working at Brookhaven, U.S.A., has performed an error analysis[8] (unpublished) on measurements made with Dietz and Goodrich[9]. Unfortunately details of the scheme were not available, however it is known that it involves partial differentials to express the sensitivity of the computed airflows to errors of measurement in the concentrations.

Two of the schemes of error analysis described above were applied to the data supplied by Dietz, and the results inter-compared.

Dietz's experiments differ from those reported above in that continuous emission of tracer gas was employed, and average, quasi steady-state concentrations were measured in each zone. Different types of tracer source were placed with one in each of three zones.

The release rates of the sources and the concentration measurements are listed in Table 6. No error estimates were provided for measurements of source rates. To solve for airflow rates, this data set can be processed in exactly the same way as for the data (eg. Table 1) of the tests above. Table 7 lists the solutions and results of the error analysis as provided.

The error norm scheme was applied by treating the standard deviations as simple errors. In applying the perturbation scheme, equation 4 was modified in the form of a sum of squared terms, and the square root of the total was taken to give the standard deviation. The matrix differentiation scheme could not be so simply converted to deal with standard deviation, and was not applied.

The results of the error analysis by perturbations are listed in Table 7. Good agreement is evident between the supplied results and those of the perturbation scheme. However, the computed error norm limits were more than an order of magnitude greater, and as such are useless.

13. VALIDATION BY RECONSTRUCTING CONCENTRATION PROFILES

Equation 1 describes a set of first order linear differential equations. Knowing the interzone airflows and the initial concentrations, it is possible to obtain the particular solutions for the concentrations in each zone as a function of time. To perform this, a computer program has previously been written[1] utilising the Runge-Kutta-Merson routine from the NAG library[11].

As an example, Figure 4 shows the reconstructed curves for both the I_{30} and G_5 solutions for Test 1, superimposed over the experimental data points obtained for the CO_2 data.

The question arises as to how far such comparisons of reconstructed curves with the original data can be used to validate the airflow solutions. We should expect the reconstructed curves to reflect the errors in the original measured quantities. Clearly the G_5 solutions are not acceptable.

However, it should be clear that the errors in the airflows represent an accumulation of the errors in taking measurements from the original data, and cannot therefore be quantified through this exersize.

13.1 Implications for Predicting Contaminant Levels

The reconstructed curves also illustrate the sensitivity of predictions of concentration profiles to errors in the airflow solutions.

In this connection we might like to use the computed airflow rates to predict the concentration levels of a contaminant which result due to a known constant source rate. We are then interested to know how sensitive are such predicted levels to errors in the airflows. An example of such a relationship between airflows and concentration levels is illustrated by Dietz's results, Table 6.

For the purposes of comparison, the errors in the airflows might be expressed in terms of the largest error divided by the largest flow, i.e. vector norms $\|\delta X\| / \|X\|$, and similarly for the errors in the concentrations, $\|\delta A\| / \|A\|$. For the results in Table 6, these quantities are 45% compared with 10% respectively. This demonstrates how relatively insensitive are the predicted concentration to errors in the airflows.

14. DISCUSSION

Error analysis showed that for the Integral Method the influence of uncertainties in the measurement of net changes in concentration were dominant. Similarly for the Gradient Method errors in the estimates of derivatives were the most important, and occasionally these could be very large. It was suggested that this Method could be improved by solving at many time points.

Error norm analysis produced large error bounds for airflows associated with zone 3 in Test 1, and uselessly pessimistic values when applied to Dietz's data. There is evidently a fundamental failing.

The underlying problem is that the elements of the matrix [A] in Test 1 are spread in value over a range of two orders of magnitude, as are those of $[A^{-1}]$ in consequence. The corresponding quantities of Dietz' data are spread over four orders of magnitute. In taking norms, i.e. maximum row sums, the influence of the smaller elements is not represented. As a result a greater accumulation of error is computed.

There is no evidence that the problem is ill-conditioned, for any of the data sets considered. Similar condition numbers (<10) were computed for both Dietz's and the test data reported here. In addition, inspection of the standard deviations and perturbations listed in Tables 5 and 7 does not suggest that the solutions were over-sensitive to changes in the input data.

The schemes of matrix differentiation and of repeated perturbation were shown to be in good agreement with each other when applied to the Test 1 data, and produced plausible results. Furthermore, the latter scheme gave similar results to an independantly proposed procedure, when applied to the same data.

The airflows and their associated errors computed using the perturbation scheme are shown for Test 1 (I_{30}) in Figure 5. The small downward airmovements from zone 3 may be due to leaks in junctions of the mechanical ventilation system situated in that zone.

It should be noted that no account has been taken of possible errors in the measurements of the effective zone volumes. The effect of any such errors can be seen by considering equation 1 for the zone flow solutions $\{X\}_{(k)}$. The differential form, analogous to equation 5, shows that an error in a zone volume (i.e. $\{dB\}_{(k)}$) produces a proportional error in all flow solutions for that zone.

15. FINAL CONCLUSIONS

Two methods of processing multitracer decay measurements to obtain interzone airflows were considered. In the Gradient Method, the derivatives of the concentration-time curves were measured. The Integral Method, in which the areas under the concentration curves are measured, was found to be the more reliable.

Although relatively simple to calculate, error norm limits were shown to be an unreliable indicator of errors in airflow solutions obtained from multitracer measurements.

The two schemes of error analysis involving differentiation in matrix form, and the sum of perturbations, produced similar and plausible results. They are ideally suited to a computer. The latter of the two was found to be the simplest to apply in practice. This scheme was shown to be in good agreement with an independantly developed procedure, in one case.

The procedure of reconstructing concentration profiles was shown to be a useful qualitative check on the airflow solutions.

Finally, an example set of results complete with error estimates, has been presented.

ACKNOWLEDGEMENTS.

Thanks are due to Dr M D A E S Perera and Dr P R Warren for their general help and advice in writing this paper. The author also acknowledges Dr Albasiny of The National Physical Laboratory for his assistance regarding the application of error analysis using vector norms.

REFERENCES

1. M D A E S PERERA, R R WALKER, O D OGLESBY, "Ventilation rates and intercell airflow rates in a naturally ventilated office building". Paper presented at the 4th AIC Conference on 'Air Infiltration Reduction in Existing Buildings' held in Switzerland on 26-28th September 1983.

2. C L LAWSON and R J HANSON , "Solving Least Squares Problems". Prentice-Hall, Englewood Cliffs, NJ (1974)

3. J M PENMAN and A A M RASHID, "Experimental determination of air-flow in a naturally ventilated room using metabolic carbon dioxide". Building and

Environment, Vol 17(6), pp253-256 (1982)

4. J H WILKINSON, "The Algebraic Eigenvalue Problem". Clarendon Press, Oxford (1965).

5. J H WILKINSON, "Rounding Errors In Algebraic Processes". HMSO, London (1963).

6. S J I'ANSON, C IRWIN and A T HOWARTH, "A multiple tracer gas technique for measuring airflows in houses". Proceedings of CIB W67 Third International Symposium.

7. J J PRIOR, C J MARTIN and J G F LITTLER, "An automatic multi-tracer gas method for following interzonal air movement". Paper HI-85-40 no.2, presented at the ASHRAE annual meeting, Honolulu HI, 1985.

8. R DIETZ and D'OTTAVIO, private communication; unpublished results of a data analysis, Brookhaven National Laboratory, NY, USA.

9. R N DIETZ, T W D'OTTAVIO and R W GOODRICH, "Seasonal effects on multi-zone air infiltration in some typical U.S. homes using a passive perfluorocarbon tracer technique". Presented at the World Congress on Heating, Ventilating, and Air Conditioning, Copenhagen, 1985.

10. M D A E S PERERA, "Review of techniques for measuring ventilation rates in multicelled buildings". Energy Conservation in Buildings - (ed. H Ehringer, G Hoyaux and P Zegers), Reidl Publishing Company, Dordrecht (1982).

11. NUMERICAL ALGORITHM GROUP, NAG manual (Mk 9), (1978).

APPENDIX A

THEORY OF ERROR ANALYSIS USING VECTOR AND MATRIX NORMS.

The following is an example of a well established procedure for the rigorous error analysis of matrix processes, as given by Wilkinson[4,5]. This involves the use of vector and matrix norms, and these are now defined.

The norm gives an assessment of the size of a vector or matrix. There are three norms in common use, defined by

$$\|X\|_p = (|x_1|^p + |x_2|^p + \ldots\ldots + |x_n|^p)^{1/p} \quad \ldots (p = 1,\ 2, \infty)$$

where $\|X\|_\infty$ is interpreted as $\max_i |x_i|$. The norm $\|X\|_2$ is the length of the vector $\{X\}$. The 'infinity' norm is implied throughout this paper. Corresponding to this vector norm, the matrix norm $\|A\|_\infty$ is defined as

$$\|A\|_\infty = \max_i \sum_j |a_{ij}|$$

Wilkinson considers the factors which determine the sensitivity of a solution of the system of linear algebraic equations

$$[A].\{X\} = \{B\} \qquad \ldots\ldots\ldots\ldots(1)$$

with respect to changes in the matrix $[A]$ and the right-hand sides $\{B\}$. These changes are represented by perturbations (which need not be small). In the discussion below, for clarity the square matrix $[A]$ is represented by A and the column vectors $\{X\}$ and $\{B\}$ are represented by X and B respectively.

If A is changed to $(A+\delta A)$ and B is changed to $(B+\delta B)$ then

$$(A+\delta A).(X+\delta X) = (B+\delta B) \qquad \ldots .(2)$$

$$(A+\delta A).\delta X = \delta B - \delta A.X$$

$$A(I+A^{-1}.\delta A).\delta X = \delta B - \delta A.X$$

$$\delta X = (I+A^{-1}.\delta A)^{-1}.A^{-1}.(\delta B-\delta A.X) \qquad ....(3)$$

By making use of the property of norms

$$\|P.Q\| \leqslant \|P\|.\|Q\| \qquad ....(4)$$

for any two vectors or matrices P and Q, it can be shown that

$$\|\delta X\| \leqslant \frac{\|A^{-1}.\delta A\|.\|X\| + \|A^{-1}.\delta B\|}{1 - \|A^{-1}.\delta A\|} \qquad ....(5)$$

provided $\|A^{-1}.\delta A\| < 1$. The terms $\|A^{-1}.\delta A\|$ and $\|A^{-1}.\delta B\|$ (norms of matrix products) can further be replaced by $\|A^{-1}\|.\|\delta A\|$ and $\|A^{-1}\|.\|\delta B\|$ (products of norms). In doing this, the inequality becomes more pessimistic.

Referring to equation (1), this result (5) provides an upper bound for the largest expected perturbation in the elements of $\{X\}_{(k)}$, due to perturbations in the elements of $[A]$ and $\{B\}_{(k)}$. In the case where the equations of mass balance are set out as n(n +1) simultaneous equations, then a single upper bound is computed for the largest expected error with regard to all of the flow solutions.

Application of the Error Norm Bound Expression

The error norm expression (5) involves the terms $\|A^{-1}.\delta A\|$ and $\|A^{-1}.\delta B\|$, which require actual sets of errors and their signs. These are not normally known; usually we have a set of possible errors, which may have either sign, and which may occur in any combination. It is therefore difficult to choose an appropriate set of perturbations $[\delta A]$ and $\{\delta B\}$.

One possibility is to assume all experimental errors occur, with the same sign. In this case the error norm bound computed can only be taken as a guide, possibly erring on either the optimistic or the pessimistic side.

A second option is to replace the above terms by the norm products $\|A^{-1}\|.\|\delta A\|$ and $\|A^{-1}\|.\|\delta B\|$, respectively. The definitions of the vector and matrix norms given above involves only the moduli of the elements, and so in this form the problem of choosing signs for the errors is avoided. The drawback is that there will be a tendency to compute an over-pessimistic limit for $\|\delta X\|$.

Condition Number

It is instructive to consider equation 5 expressed in terms of relative errors:

$$\frac{\|\delta X\|}{\|X\|} \leqslant \frac{\|A\|.\|A^{-1}\|.\left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta B\|}{\|B\|}\right]}{1 - \|A\|.\|A^{-1}\|.\frac{\|\delta A\|}{\|A\|}} \quad \ldots.(3)$$

where the norm products have been substituted throughout, and use has been made of the properties of norms (4) to replace $\|X\|$ by $\|b\|/\|A\|$ on the right hand side.

It can be seen that a decisive quantity is $\|A\|.\|A^{-1}\|$. This expresses the sensitivity of the solution to perturbations in the parameters, and is termed a 'condition number' for the problem. Ill-conditioning is indicated by $\|A\|.\|A^{-1}\|$ much greater than unity.

APPENDIX B

The equations governing the continuity of air and three tracer gases in three zones are set out as follows

$$[A].\{X\} = \{B\}$$

$$\begin{bmatrix}
C01 & 0 & 0 & -C11 & -C11 & -C11 & 0 & C21 & 0 & 0 & C31 & 0 \\
C03 & 0 & 0 & -C12 & -C12 & -C12 & 0 & C22 & 0 & 0 & C32 & 0 \\
C03 & 0 & 0 & -C13 & -C13 & -C13 & 0 & C23 & 0 & 0 & C33 & 0 \\
1 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & C01 & 0 & 0 & C11 & 0 & -C21 & -C21 & -C21 & 0 & 0 & C31 \\
0 & C02 & 0 & 0 & C12 & 0 & -C22 & -C22 & -C22 & 0 & 0 & C32 \\
0 & C03 & 0 & 0 & C13 & 0 & -C23 & -C23 & -C23 & 0 & 0 & C33 \\
0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 \\
0 & 0 & C01 & 0 & 0 & C11 & 0 & 0 & C21 & -C31 & -C31 & -C31 \\
0 & 0 & C02 & 0 & 0 & C12 & 0 & 0 & C22 & -C32 & -C32 & -C32 \\
0 & 0 & C03 & 0 & 0 & C13 & 0 & 0 & C23 & -C33 & -C33 & -C33 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & -1
\end{bmatrix}
\times
\begin{Bmatrix}
Q01 \\ Q02 \\ Q03 \\ Q10 \\ Q12 \\ Q13 \\ Q20 \\ Q21 \\ Q23 \\ Q30 \\ Q31 \\ Q32
\end{Bmatrix}
=
\begin{Bmatrix}
V1.C'11 \\ V1.C'12 \\ V1.C'13 \\ 0 \\ V2.C'21 \\ V2.C'22 \\ V2.C'23 \\ 0 \\ V3.C'31 \\ V3.C'32 \\ V3.C'33 \\ 0
\end{Bmatrix}$$

Where

CIJ = Concentration in Ith zone, of Jth tracer
QIJ = Airflow from zone I to zone J; zone 0 refers to the outside air mass
VJ = Volume of zone J
C'IJ = Time derivative of CIJ

To obtain the equivalent expression for the equations in integral form, replace CIJ by $\int_{\tau}$CIJ.dt, and C'IJ by ΔCIJ (nett change in concentration over the period τ.

ZONE:	1	2	3	1	2	3
VOLUME:	93.5	81.5	260 m^3			
	$V_j\Delta C_{ij}$: m^3 xconc			$\int_\tau C_{ij}.dt$: conc.xmin		
CO_2 (1)	-3179 $\pm$ 187	-1060 $\pm$81.5	-260 $\pm$260	1098	696	120
N_2O (2)	-3833 $\pm$93.5	-4971 $\pm$ 163	-2340 $\pm$ 260	1910	2708	854
SF_6 (3)	0 $\pm$93.5	- 326 $\pm$ 163	-15080 $\pm$ 5020	377	585	4987
				estimated error $\pm$ 7 concxmins		

200 conc. units = 200 ppm SF_6/N_2O ; 5000ppm CO_2

Table 1 : Integral Method, Test 1. Measured quantities evaluated over 30 minutes

ZONE:	1	2	3	1	2	3
VOLUME:	93.5	81.5	260 m^3			
	$V_j.\dot{C}_{ij}$: m^3 xconcxmin^{-1}			C_{ij} : conc.units		
CO_2 (1)	- 119 $\pm$ 10	- 33 $\pm$ 6	- 16 $\pm$ 5	41.5	26	4.5
N_2O (2)	- 105 $\pm$ 19	-161 $\pm$ 24	-101 $\pm$ 21	72.5	103.5	31.5
SF_6 (3)	0 -	0 -	-629 $\pm$109	13.5	21.5	186
				estimated error $\pm$ 1 conc.unit		

200 conc . units = 200ppm SF_6/N_2O ; 5000ppm CO_2

Table 2 : Gradient Method, Test 1. Measured quantities evaluated at 11 minutes elapsed time

	Integral Method		Gradient Method	
	I_{30}	I_{15}	G_{11}	$*G_5$
Q_{10}	188	162	223	272
Q_{20}	62	113	0	-10
Q_{30}	164	152	187	83
Q_{01}	142	149	128	178
Q_{21}	64	79	104	108
Q_{31}	9	8	5	7
Q_{02}	97	102	79	89
Q_{12}	23	28	21	33
Q_{32}	10	11	11	13
Q_{03}	175	176	202	78
Q_{13}	4	47	6	-12
Q_{23}	3	-51	7	38

m^3 /hr

* suspected underestimation of zone 3 N_2O derivative

Table 3 : Test 1 airflow results by all methods

	Integral Method		Gradient Method	
	I_{30}	I_{15}	G_{11}	G_5
Q_{10}	223	242	164	271
Q_{20}	-2	-4	40	-2
Q_{30}	323	334	256	357
Q_{01}	148	159	131	174
Q_{21}	124	132	93	138
Q_{31}	6	9	1	12
Q_{02}	115	118	116	122
Q_{12}	71	71	87	66
Q_{32}	14	13	13	14
Q_{03}	281	295	213	330
Q_{13}	-16	-13	-26	-13
Q_{23}	78	74	83	66

m^3 /hr

Table 4 : Test 2 airflow results by all methods

	Means; Test 1	Std.Devn. measured	Norm.limit	Pertbns	Differn.
Q_{10}	191	25		73	77
Q_{20}	58	46		67	71
Q_{30}	168	14		10	12
Q_{01}	140	9		9	10
Q_{21}	82	16	36	20	21
Q_{31}	7	2		2	2
Q_{02}	93	10		6	6
Q_{12}	24	3	28	14	15
Q_{32}	11	<1		3	1
Q_{03}	184	12		17	17
Q_{13}	19	20	67	36	36
Q_{23}	-14	26		34	34
			m^3/hr		

Table 5 : Mean and standard deviation of Test 1 results, and error estimates using I_{30} data Test 1

ZONE:	1	2	3		1	2	3
VOLUME:	255	250	123	m^3			
	SOURCE RATE nl/hr				AVERAGE CONC. nl/m^3		
GAS:							
PDCH (1)	1319	-	-		19.22 *(1.5)	9.12 (2.39)	0.17 (0.02)
PMCH (2)	-	3045	-		23.51 (0.29)	24.99 (0.39)	0.44 (0.06)
PDCB (3)	-	-	1153		9.08 (0.19)	9.63 (0.23)	25.3 (4.29)
(no error estimate supplied)					*( ) standard deviations		

Table 6 Measured data, continuous source method (Dietz et al)

	Dietz/D'Ottavio airflows	Std. Devn.	Pertbns $\sqrt{\delta f^2}$	norms $\|\delta X\|$
Q_{10}	20	28	30	
Q_{20}	103	33	28	
Q_{30}	-1	7	2	
Q_{01}	7	3	3	
Q_{21}	117	32	35	388
Q_{31}	<1	2	2	
Q_{02}	70	11	8	
Q_{12}	104	53	64	690
Q_{32}	47	10	7	
Q_{03}	45	9	7	
Q_{13}	<1	<1	1	115
Q_{23}	1	<1	1	
	m^3/hr			

Table 7 : Results of error analysis using perturbations and error norm bounds, compared with independant scheme.

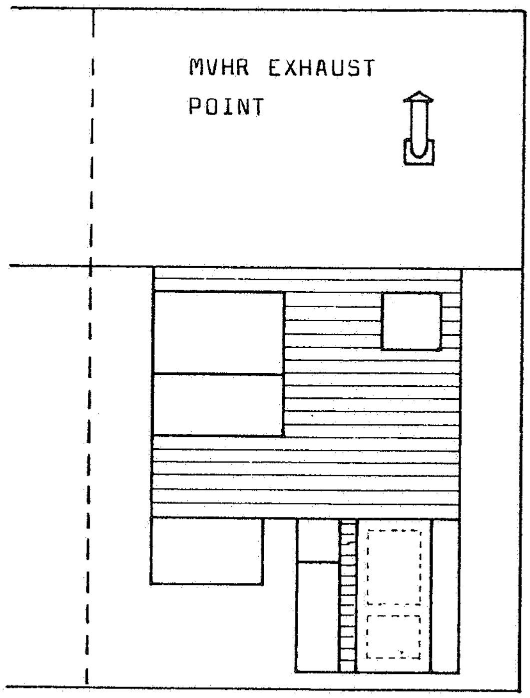

FIGURE 1. TEST HOUSE: NORTH ELEVATION

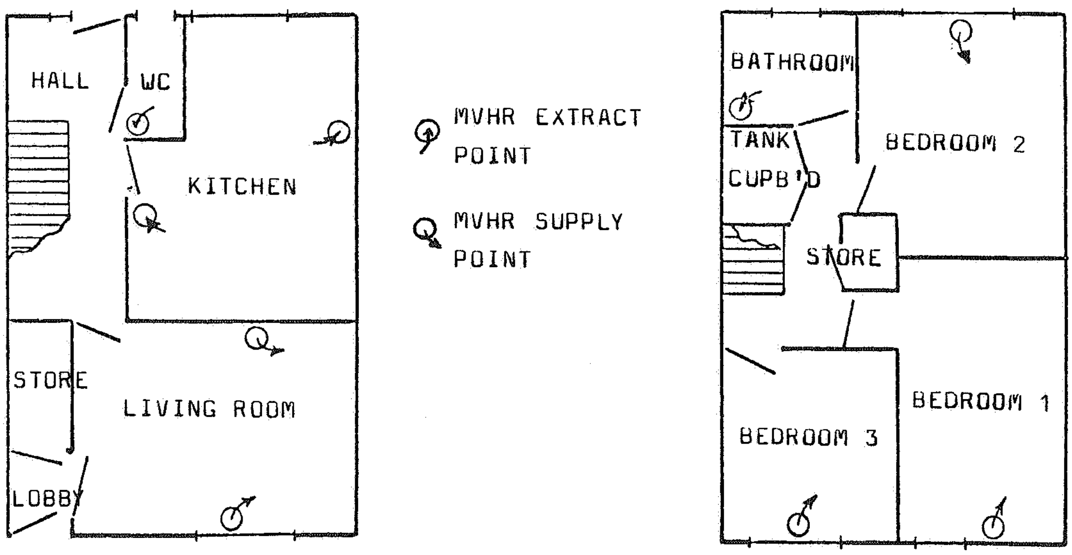

FIGURE 2. TEST HOUSE: FLOOR PLANS

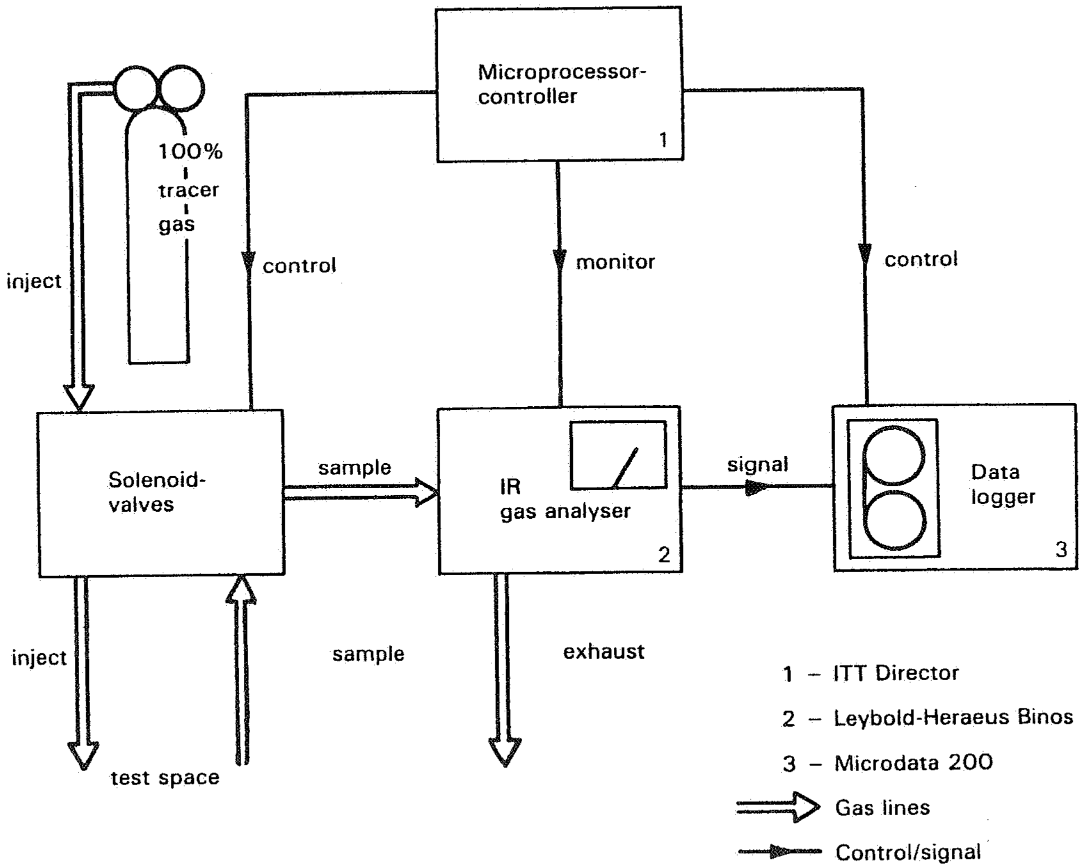

FIGURE 3. Microprocessor-controlled ventilation rate measuring system (Tracer gas decay)

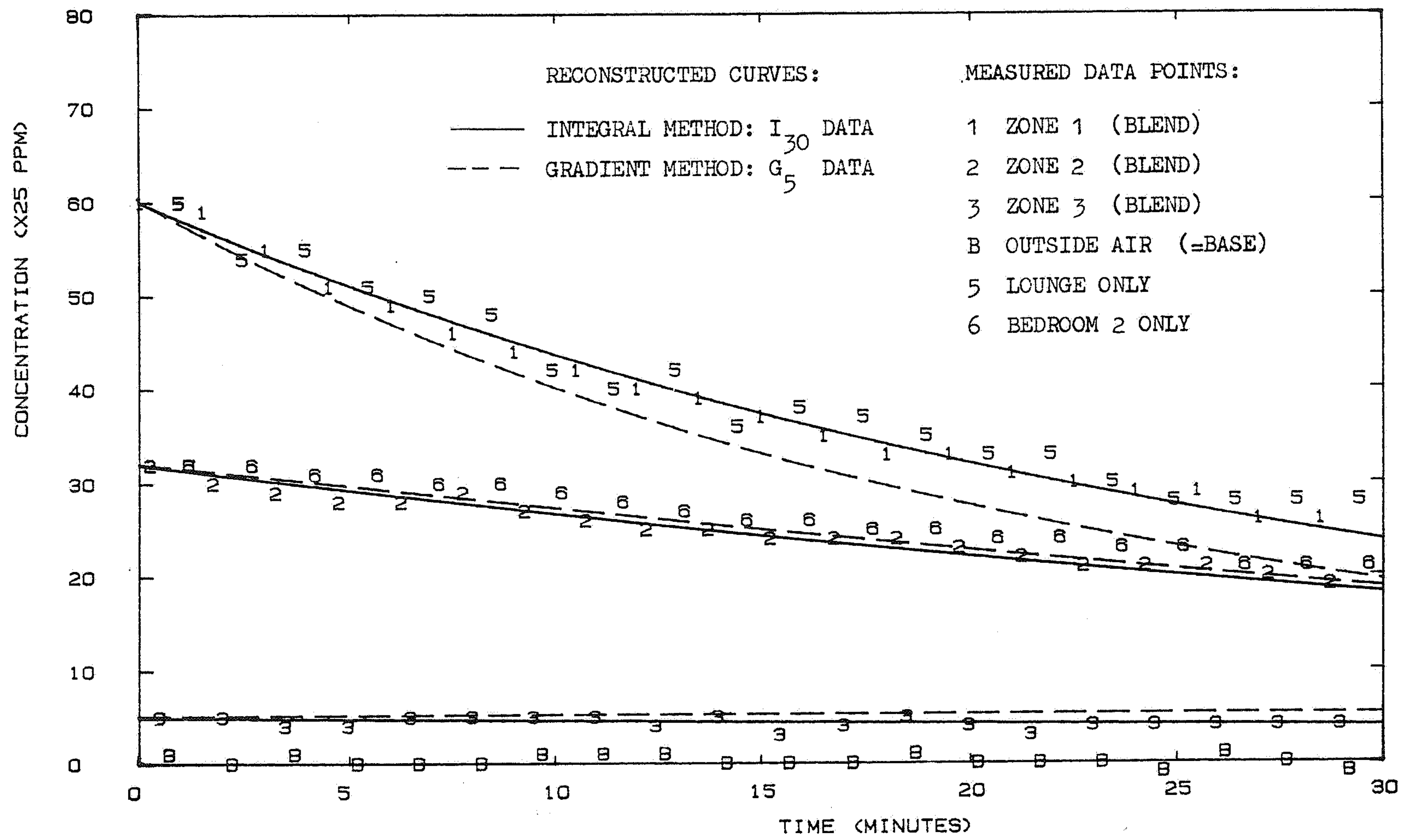

FIGURE 4. Test 1 CO_2 - measured data points and reconstructed curves

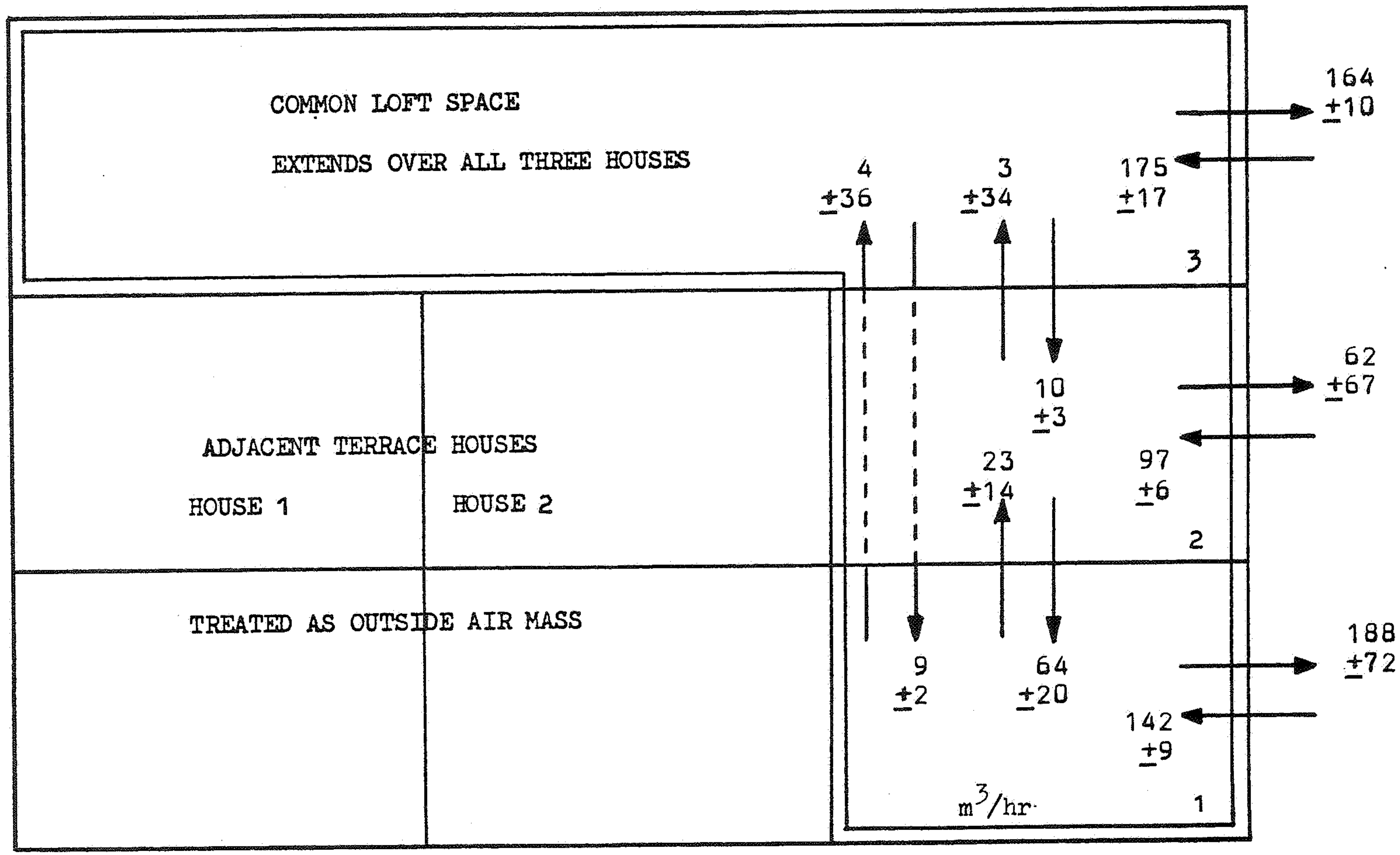

FIGURE 5. Airflows and error estimates Test 1: Integral method (I_{30} Data)