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Heat Loss from a Solid Ground Floor

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Heat losses from solid uninsulated ground floors are often calculated using Macey's formula which involves the floor length and breadth, and the thickness of the wall surrounding it. The origins of the formula are examined. It is shown that in deriving it, an assumption is implicitly made that a semi-circular section of the floor beneath the wall is composed of perfectly insulating material. Further, the adjustment which is used to make the formula apply to a floor of finite length leads to an internally inconsistent expression. Procedures are advanced that tend to correct for both these defects. An exact expression has been advanced by Delsante, Stokes and Walsh against which five simplified forms can be tested and one of them is compact and provides 1% accuracy.

1. INTRODUCTION

THE U value of a building construction is defined as the heat loss through it per unit area for one degree difference in temperature between the room index temperature on one side and ambient temperature on the other. It includes consideration of the films which describe the convective and radiative processes which bring heat from the room interior to the inner surface of the construction and remove it at the exterior surface. (Typical film resistances are 0.12 m^2 K/W inside and 0.06 m^2 K/W outside.) The U value is taken to be the same over all areas which have the same formation.

This definition applies to walls and roofs. In order to find losses through solid floors, a U value is defined based on the temperature of floor itself, and also the surface temperature of the ground outside. Thus it ignores the film resistances. (The omission usually makes little difference.) However, the heat loss per unit area from a floor varies with position: it has its largest value at the perimeter wall and its least value in the middle of the room. Since the designer is concerned with a total heat loss Q from a floor rather than with its U value *per se*, this discussion will be drafted in terms of Q rather than U.

The heat loss from a rectangular floor is clearly proportional to the temperature difference ΔT between the inside and outside horizontal surfaces and to the conductivity λ of the soil. It must further depend in some way upon the length L, breadth B and wall thickness W, as shown in Fig. 1. Thus we can write

$$Q = \lambda \cdot \Delta T \cdot G, \tag{1}$$

where G is some geometrical function of L, B and W. It has the units of length. The question discussed in this paper is: what function should we choose for G?

The function provided in the current CIBSE Guide [1], is

$$G = \frac{2}{\pi} L \ln\left[\frac{2B}{W}+1\right] \cdot \exp\left[\frac{B}{2L}\right]. \tag{2}$$

(This is based on [1], equations (A3.26) and (A3.27). The Guide uses an arctanh function but as Anderson [2] has pointed out, it can be equally expressed as a log function and this is the more convenient form.) This form for G was derived by Macey [3] in 1949. Examination of the expression and its origins however shows that it tacitly presupposes a section of pure insulating material—a non-existent entity—in the floor construction, and further that the expression is internally inconsistent.

Another form for G was published in 1983 by Delsante, Stokes and Walsh [4]:

$$G = \frac{2}{\pi}\left[L \ln\left[\frac{2L}{W}\right] + B \ln\left[\frac{2B}{W}\right] + 2(L^2+B^2)^{1/2} - L - B - B \ln\frac{(L^2+B^2)^{1/2}+B}{L} - L \ln\frac{(L^2+B^2)^{1/2}+L}{B}\right]. \tag{3}$$

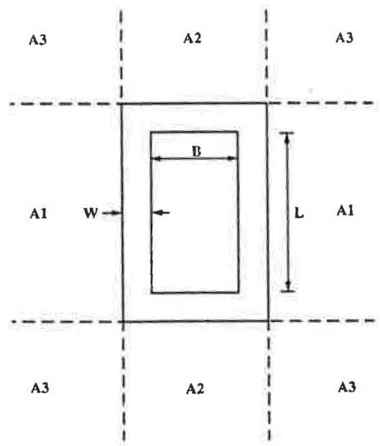

Fig. 1. The model for loss of heat from a room through a solid uninsulated floor to the exterior. The internal floor area $L \times B$ is maintained at T_i and heat flows through the floor material under the area occupied by the wall (thickness W) to the surrounding area at T_o.

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The Delsante/Stokes/Walsh expression is based upon an exact solution of the Laplace equation (equation of heat continuity) in the ground,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial x^2}=0 \tag{4}$$

provided in Carslaw and Jaeger [5]. (Delsante *et al.* give a 17-term expression for G. If each term is expanded into terms in W^{-1}, W^0 and W^1... and only terms in W^{-1} and W^0 are retained, it reduces to equation (3). This is justified when $L \gg W$ and $B \gg W$, which is of course normally the case.)

The consequences of the Delsante/Stokes/Walsh expression are not in fact dramatically different from those based on that of Macey so the Guide expression is not misleading. Indeed, the Guide values are based on an assumed value of λ equal to 1.4 W/mK, but as it is pointed out there, λ may be between 0.7 and 2.1 W/mK depending on soil conditions. Uncertainty in λ rather than the handling of floor geometry is the more likely source of uncertainty, and there are other factors which neither expression can take account of.

As far as floor geometry goes, the Delsante/Stokes/Walsh expression is clearly superior to that of Macey, is computationally straightforward and so might replace the earlier expression. In view of the long standing of the Macey formula however, it would seem appropriate that its defects were demonstrated, and that attempts to remedy them should be reported. The remainder of this article is concerned with these issues. Rather surprisingly, they lead to a form for heat loss which is simpler and more accurate than the approximate form of Delsante *et al.*

2. DERIVATION OF MACEY'S FORMULA

Macey considered in the first instance a rectangular floor of indefinite length L and finite breadth B with walls, each of thickness W, outside the breadth dimension. (These are notated by Macey as D and $2R-d$ and d respectively.)

2.1. *Use of conjugate functions*

Macey based his analysis on the treatment by Carslaw [7] Section 113 which deals with the use of conjugate functions in two-dimensional heat flow problems. If α and β are real functions of x and y such that

$$\alpha+i\beta = f(x+iy) \tag{5}$$

and v is some function of x and y that satisfies the Laplace equation in two dimensions:

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \tag{6}$$

then this is true too for the transformed coordinates:

$$\frac{\partial^2 v}{\partial \alpha^2}+\frac{\partial^2 v}{\partial \beta^2}=0. \tag{7}$$

More specifically, if the function is

$$\alpha+i\beta = \ln\frac{x+1+iy}{x-1+iy} \tag{8}$$

it appears that

$$\alpha = \ln\frac{r_2}{r_1} \quad \text{and} \quad \beta = \theta_2-\theta_1, \qquad \text{(9a) and (9b)}$$

where r_1 is the distance between the field point (x, y) and the axial point $(+1, 0)$, θ_1 is the angle r_1 makes with the positive direction of x, r_2 is the distance between (x, y) and $(-1, 0)$ and θ_2 is the angle between r_2 and positive x. (Private correspondence with Dr Delsante regarding equations (8) and (9) leads me to point out that both Carslaw and Macey give equation (9b) as $\beta = \theta_1-\theta_2$.)

If the field point (x, y) moves in such a way that β remains constant, it traces out a circle—a well known property of circles. If the point moves so that α is constant, it also traces out a circle. (This is not so obvious; it is demonstrated in an appendix.) Thus the relations α = constant and β = constant define families of circles; large circles of the constant α family enclose the smaller circles but do not intersect them; circles of the constant β family pass through the points $(+1, 0)$ and $(-1, 0)$; circles of different families intersect each other at right angles—the orthogonal property.

Macey picked up Carslaw's argument at this point. He identified lines of constant α as the directions of heat flow lines, lines of constant β as isothermal lines and made very ingenious use of the construction. As his Fig. 1 indicates, these lines would be generated in a semi-infinite conducting medium of indefinite length L, finite breadth B and $W = 0$. The temperature over the breadth B is supposed to be at some uniform value T_1 say and the temperature everywhere outside it to be zero. The heat flow lines (or more exactly, cylindrical surfaces), therefore represent the loss of heat from the area $L\times B$ to the area outside it and the isotherms are the cylindrical surfaces linking points at the same temperature at various values between T_1 and zero.

Macey recognized that there would be an infinite temperature gradient if $W = 0$. He therefore assumed that '... while the presence of the finite wall prevents that heat flow which in the ideal case ($W = 0$) would have taken place under its foundations, it does not otherwise affect the isothermals and lines of flow. In other words, the flow that takes place with a finite wall is that which occurs in the ideal case through the same amount of uncovered earth ...' This seems a fair assumption as far as it goes, but the analysis takes no account of what, if anything, happens in the *covered earth.* What indeed constitutes the covered earth? Is it of finite or infinite depth? Since this is the nub of the matter, a different approach to finding Macey's expression will be presented; it makes clear just what 'covered earth' means.

2.2. *An alternative basis for the heat loss formula*

It is convenient to provide an alternative basis for the heat loss formula to that given by the conjugate transformation. It uses elementary considerations of heat flow.

We start by considering a line heat source of indefinitely large length, embedded horizontally and centrally in a medium of conductivity λ and large circular section (radius R) whose boundaries are at zero temperature. The cylinder loses q W per meter run. The heat flow across a notional cylindrical surface of radius r, and concentric with the cylinder, is

$$q = -2\pi r\lambda \cdot \frac{dT}{dr} \quad \text{or} \quad -2\pi\lambda \cdot \frac{dT}{dr/r}. \tag{10}$$

This integrates directly, so we can express the temperature T_{m1} at some point in the medium distant r_1 from the line source as

$$T_{m1} = -\frac{q \ln (r_1/R)}{2\pi\lambda}. \tag{11}$$

Suppose now that a second line source is laid parallel to the first and distant $2d$ from it in the same horizontal plane. It 'loses' $-q$ W per meter run (and thus serves as a sink for the heat output of the first source). The temperature T_{m2} at a point in the medium distant r_2 from the centre of the second is given by an expression similar to that above.

The temperature T_m due to both cylinders is $T_{m1} + T_{m2}$, so

$$T_m = \frac{q \ln (r_2/r_1)}{2\pi\lambda}. \tag{12}$$

T_m must be positive for a point P which is nearer the first cylinder, and is zero anywhere on the vertical central plane. If P moves through the medium in such a way that the ratio r_2/r_1 remains constant, P moves along a circle, as will be shown in the Appendix. Thus the relation r_2/r_1 = constant describes isothermal cylindrical surfaces. See Fig. 2a. The isotherms however are not concentric with the line source they contain: if D denotes the distance of the centre of an isotherm from the central plane, and c its radius, it is found that

$$D^2 - c^2 = d^2. \tag{13}$$

The lines indicating the direction of the flow of heat from one line source to the other can also be traced out and they too prove to be circular. They pass through the line sources.

Thus using the line source/sink system we have arrived at the two families of circles already found using the conjugate function. The physical model that generates them however—two parallel line sources—is not appropriate and two qualitative transformations are needed.

(a) *Bounding surfaces.* Consider a limited region of the flow field—the region below the horizontal for example. It is a principle that temperature and flow in this limited region of the medium remain unaltered if the limited region

(i) is terminated by a perfectly conducting surface held at some fixed temperature, which coincides with an isothermal surface of that temperature in the parent system, or
(ii) is terminated by a perfectly insulating surface which coincides with a heat flow surface in the parent system.

These principles are self-evident. See Fig. 2b. We replace the line sources by cylinders of finite diameter

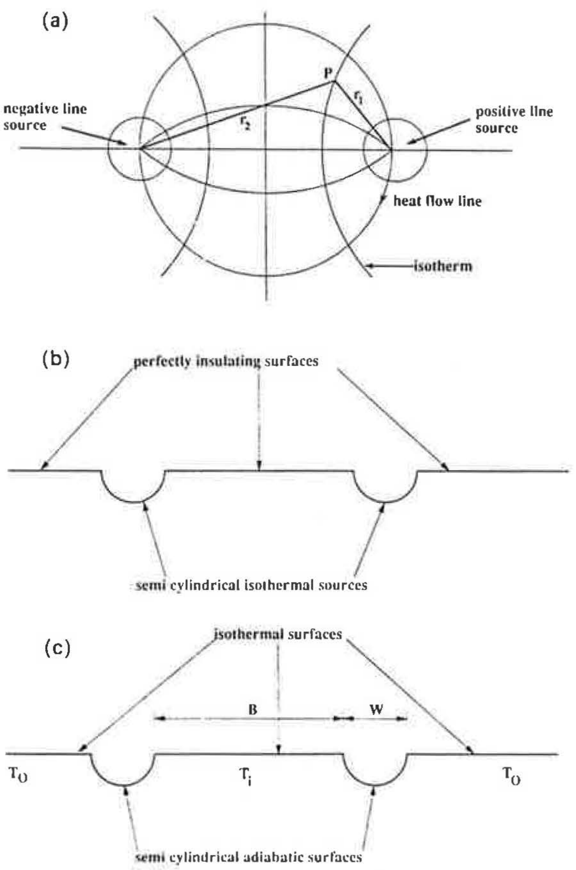

Fig. 2. Macey's model for heat loss through a solid floor between isothermal horizontal surfaces separated by a gap. (a) Isotherms and heat flow lines due to a pair of parallel and equal line sources. (b) The model when the flow field is limited by isothermal and adiabatic surfaces. (c) The building model which results when the surfaces of (b) reverse their identities.

(emitting $\pm q$ W/m) which coincide with isothermal surfaces and the heat flow in the conducting medium remains the same elsewhere. It is to such cylinders rather than line sources that reference will be made from now on. The volume within the cylinders does not form part of the flow field.

Further, the flow field below the horizontal remains unaltered if the horizontal surface between the cylinders and the two infinitely extended horizontal surfaces outside the cylinders—both flow lines—are replaced by perfectly insulating surfaces and the upper volume of conducting medium is removed.

(b) *Reversal of identities.* A further transformation is needed. Since the isotherms and flow lines form an orthogonal system, they can reverse their identities: what was formerly an isotherm can be interpreted as a flow line and what was formerly a flow line can be interpreted as an isotherm.

Applying this idea to Fig. 2b we have Fig. 2c. The horizontal flow line between the cylinders becomes an isothermal surface—the floor surface at a positive temperature T_i and the two infinitely extended horizontal surfaces outside the cylinders become isothermal surfaces representing the ground temperature T_0. The two former source-sink cylinders become semi-cylinders below the horizontal, and since they were formerly isothermal, they become *adiabatic surfaces*, and must be composed of perfectly insulating material if we suppose they are solid. Isothermal surfaces of circular section can be located in the field between the limiting values of T_i and T_o.

2.3. *Macey's analysis*

The remainder of the discussion follows Macey's analysis. We want to find the heat flow from the area of indefinite length L and breadth B between the cylinders at T_i through the conducting medium to the ground surface at T_o. The heat flow through the elementary area $L \cdot dx$ is

$$dQ = -\lambda L \cdot dx \frac{dT}{dy} \text{ vertically downward.} \quad (14)$$

Now dT/dy is the rate of penetrating isothermal surfaces, and in this problem the isothermal surfaces are, very conveniently, circular in section. Consider the point $P(x, \delta y)$ on the isotherm (Fig. 3) a distance δy just below the inside surface and join P to the equivalent line source locations so as to make the angle β as shown. The relation, $\beta =$ constant, defines a circle and so an isotherm. The gradient can be expressed as

$$\frac{dT}{dy} \equiv \frac{dT}{d\beta}\frac{d\beta}{dy}. \quad (15)$$

Now the isotherms are spaced uniformly around the locations of the former line sources. Thus as we move from a field point P on the inner surface where $\beta = \pi$ and $T = T_i$ to a point on the outer surface where $\beta = 0$ and $T = T_o$ we have

$$\frac{dT}{d\beta} = \frac{T_i - T_o}{\pi}. \quad (16)$$

Further, the angle β between these two directions is

$$\beta = \pi - \left[\frac{\delta y}{d-x} + \frac{\delta y}{d+x}\right] \quad (17)$$

so

$$\frac{d\beta}{dy} = -\frac{2d}{d^2 - x^2}. \quad (18)$$

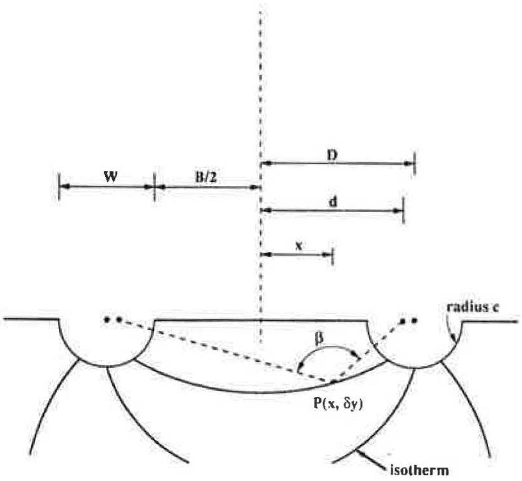

Fig. 3. To illustrate the derivation of Macey's formula.

Thus the total heat flow over the floor of width b is

$$Q = \frac{-\lambda L \cdot 2d \cdot (T_i - T_o)}{\pi} \cdot \int_{-B/2}^{B/2} \frac{-dx}{d^2 - x^2}. \quad (19)$$

From the standpoint of this idealized model, the semi-cylinders (radius c or diameter W) simply serve to separate the room floor area at T_i from the surrounding land areas at T_o. From the building point of view, these areas are separated by a wall which sets up this temperature difference, at any rate approximately. The value for the cylinder diameter therefore is taken to be the wall thickness (although thermal aspects of the wall are irrelevant as such).

It is evident from Fig. 3 that the centre of the cylinder is to be located at

$$D = B/2 + c = B/2 + W/2. \quad (20)$$

Further, the fictitious distance d appearing in equation (19) is given from equation (13) as

$$d^2 = D^2 - (W/2)^2. \quad (21)$$

Then

$$Q = \lambda(T_i - T_o) \cdot \frac{2}{\pi} \cdot L \ln\left[\frac{\sqrt{(1 + 2W/B)} + 1}{\sqrt{(1 + 2W/B)} - 1}\right]. \quad (22)$$

This appears to be the exact form for the heat loss from a length L in an effectively infinitely long floor, according to the cylinders model. If the room width is large in relation to wall thickness, as is normally the case, so that $W/B \ll 1$

$$Q = \lambda(T_i - T_o) \cdot \frac{2}{\pi} \cdot L \ln\left[\frac{2B}{W} + 1\right] \quad (23)$$

or

$$Q = \lambda(T_i - T_o) \cdot \frac{4}{\pi} \cdot L \operatorname{arctanh}\left[\frac{B}{B + W}\right] \quad (24)$$

which apart from notation is the original Macey form for a very long floor. If $W/B \gg 1$,

$$Q = \lambda(T_i - T_o) \cdot \frac{2}{\pi} \cdot L\left[\frac{2B}{W}\right]^{1/2}. \quad (25)$$

It is clear that Macey's expression is an approximation to an exact result that it is valid only if the heat flow line starting from the floor at the wall itself follows a semi-circular path to the ground outside, rather than the more direct path that it must follow. Thus the analysis tactitly assumes the presence of a semi-cylinder of perfectly insulating material in the floor under the wall. This constitutes the 'covered earth' implied in Macey's assumption stated earlier. His assumption however does not imply a step change in surface temperature as has been suggested but rather a gradual fall round the semicircular surface.

In the following sections, we shall address the questions:

(i) How may we correct for the inclusion of a semi-cylindrical volume of perfectly insulating material? (Section 3.)

(ii) Inclusion of the factor $\exp(B/2L)$ to correct for the finite length L of a slab leads to an internally inconsistent expression. Can the expression be better generalized? (Section 4.)
(iii) What penalty may there be in ignoring the inside and outside film coefficients that are normally included in conductive heat loss calculations? (Section 5.)

3. CORRECTION FOR INCLUSION OF AN ADIABATIC SEMI-CYLINDER BENEATH THE WALL

An adiabatic semi-cylinder of material in a floor slab under a wall must lead to an underestimate of the heat loss. The correct loss can be found using Fourier analysis as shown below and this enables us to include a correction to the Macey expression.

For the present purpose, conduction in the slab to the right hand side of the floor centre line alone will be considered. The conduction model for floor loss thus involves a quarter-infinite slab of conducting material, adiabatic at the vertical plane through the floor centre, with a horizontal surface consisting of two isothermal areas separated by a strip of width W in which there is a uniform fall in temperature from T_i to T_o. This will be simplified to the section of two dimensional conducting material shown in Fig. 4. The shape is rectangular of width a and depth c. Three of its edges are adiabatic and on the fourth is imposed a temperature distribution

$$T(1,x,0) = T_1 \cos(\pi x/a). \tag{26}$$

It is easily checked that the temperature distribution in the medium is given by

$$T(1,x,y) = \frac{T_1}{\cosh\frac{\pi c}{a}}\cos\frac{\pi x}{a}\cosh\frac{\pi(c-y)}{a}. \tag{27}$$

It satisfies the boundary conditions and also the continuity equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{28}$$

The heat input $q(1,x,0)$ at the surface is given by $-\lambda(\partial T(1,x,0))/\partial y$

$$q(1,x,0) = \lambda \cdot T_1 \tanh\frac{\pi c}{a}\cos\frac{\pi x}{a}. \tag{29}$$

All heat which enters at the top left edge leaves again in the top right and two heat flow lines are indicated.

Further solutions can be constructed: $T(n,x,y)$ has the same form as the above equation with T_n replacing T_1 and a/n replacing a. The temperature distribution shown in Fig. 4b can then be set up by superposition of such solutions using standard Fourier analysis:

$$T(x,0) = \sum T(n,x,0) = T_i[\tfrac{1}{2}c_0 + \sum c_n \cdot \cos n\pi x/a] \tag{30}$$

summing n from 1 to infinity where

$$c_0 = \frac{1}{a}(\tfrac{1}{2}B + \tfrac{1}{2}W) \tag{31a}$$

and

$$c_n = \frac{4}{(n\pi)^2}\frac{a}{W}\left[\sin\frac{n\pi\frac{1}{2}W}{a}\cdot\sin\frac{n\pi(\frac{1}{2}B+\frac{1}{2}W)}{a}\right]. \tag{31b}$$

The total local heat input at the surface is $\Sigma\, q(n,x,0)$ and the heat input over the floor area of semi-width $\frac{1}{2}B$ is found by integration.

$$Q = \lambda(T_i - T_o)\cdot L\sum\left[c_n \tanh\frac{n\pi c}{a}\cdot\sin\frac{n\pi\frac{1}{2}B}{a}\right]. \tag{32}$$

Since the room to be modelled is supposed to be placed upon earth of unrestricted extent and depth, a and c must be given large numerical values in these expressions. It is convenient to non-dimensionalize Q by division by $\lambda(T_i - T_o)L$ and we define ϕ_F, the Fourier version of the heat loss, as

$$\phi_F = \sum\left[c_n \tanh\frac{n\pi c}{a}\cdot\sin\frac{n\pi\frac{1}{2}B}{a}\right] \tag{33}$$

and the ϕ_M, the modified Macey version (for one-sided loss) is given as

$$\phi_M = \frac{1}{\pi}\ln\left[\frac{\sqrt{(1+2W/B)}+1}{\sqrt{(1+2W/B)}-1}\right]. \tag{34}$$

These quantities were evaluated for the range of full

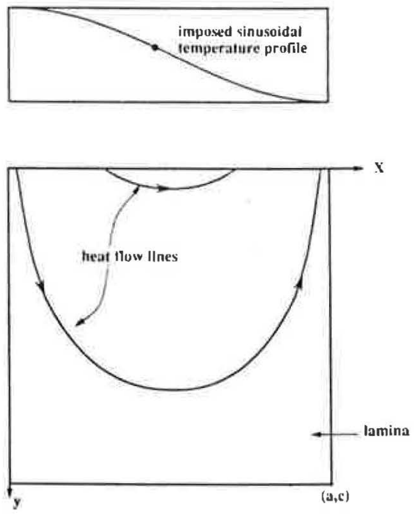

Fig. 4(a). A rectangular lamina with three adiabatic sides and a sinusoidally varying temperature distribution imposed on the fourth. The arrowed lines show the direction of heat flow.

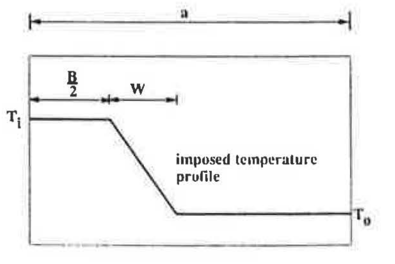

Fig. 4(b). The temperature imposed upon the lamina so as to approximate to that used in the building model.

Table 1. Heat losses from one side of a solid isothermal uninsulated floor, semi-width $B/2$ and of infinite length past a gap width W to an isothermal large land area. ϕ_F denotes the value corresponding to unimpeded flow past the gap; ϕ_M denotes the value when the gap is occupied by a semi-cylinder of perfectly insulating material

	W					
	0.2 m		0.3 m		0.4 m	
B (m)	ϕ_F	ϕ_M	ϕ_F	ϕ_M	ϕ_F	ϕ_M
100	2.29	2.20	2.17	2.07	2.08	1.97
60	2.13	2.04	2.00	1.91	1.91	1.82
40	2.01	1.91	1.88	1.78	1.79	1.69
20	1.79	1.69	1.66	1.56	1.57	1.47
10	1.57	1.47	1.44	1.35	1.35	1.26
6	1.41	1.31	1.28	1.19	1.19	1.10
4	1.28	1.19	1.15	1.07	1.07	0.98
2	1.07	0.98	0.94	0.87	0.86	0.79
Mean difference	0.093		0.092		0.090	

widths (B) in the current CIBSE Guide, Table A3.10. and values of $W = 0.2$, 0.3 and 0.4 m. See Table 1.

The overall mean difference $\phi_F - \phi_M$ is about 0.091 and there is little variation about it. Replacing ϕ_M by $\phi_M + 0.091$, a corrected value for the heat loss can be written as

$$Q = \lambda(T_i - T_o) \cdot L \frac{2}{\pi} \cdot \left[\ln \left[\frac{\sqrt{(1+2W/B)}+1}{\sqrt{(1+2W/B)}-1} \right] + 0.091 \times \pi \right]. \quad (35)$$

3.1. *Note on computation*

The length a used in the Fourier computation consisted of the floor semi width $B/2$, the gap width W through which the temperature was supposed to fall uniformly, and the land area beyond the wall. The Macey approach assumes of course that the land is of infinite extent. a cannot be made infinite in the Fourier approach: it was expressed as a/B and took typically large values. W was handled as W/B, and was typically small. The Fourier series was evaluated by fixing W/B and increasing a/B until no substantial increase in ϕ_F took place. Table 2 shows an example.

ϕ_F at first increases a little with a/B, corresponding to the increase in conductance from a given floor/wall combination when the heat is able to spread to a larger surface. We expect ϕ_F to converge to a steady value. In fact the series converges very slowly especially for higher values of a/B. Values stable to 2 decimal places were found with values of a/B of 100 or 200 and summing up to some 300 000 terms. (Values of ϕ_F apparently decrease with increase of a/B; sufficient terms must be included to reach a stable value.) A value of $a/B = 100$ is clearly physically reasonable: with a floor width of 10 m, a one-sided land area of 10×100 or 1000 m is effectively infinite.

Table 2. Values of ϕ_F with increasing land width a. $B = 6$ m, $W = 0.3$ m

a/B	1	2	5	10	20	50	100	200
ϕ_F	1.126	1.243	1.274	1.278	1.279	1.280	1.280	1.279

According to an expression in Section 3.3 of [4] and also as equation (11) in [9], ϕ_F is directly calculable as

$$\phi_F = \frac{1}{\pi} \ln \left[(1+x)\left(1+\frac{1}{x}\right)^x \right] \quad \text{where} \quad x = \frac{B}{W}. \quad \text{(36a) and (36b)}$$

This is a more efficient method, but the Fourier approach readily allows estimation of other quantities such as the distribution of the heat flux into the slab and the pattern of isotherms and heat flow lines.

3.2. *Comment*

ϕ_M and ϕ_F have the same basis to the extent that in each case a floor of semi-width $B/2$ and a land area forming isothermal surfaces is assumed. The gap width however is treated differently. In finding ϕ_M, the space occupied by W is adiabatic and the temperature distribution follows from this: a gradual fall over a semi-circular surface. In finding ϕ_F, we assume a uniform temperature fall over the plane surface but this is only possible if there is a heat input, positive and negative over the width W.

The latter assumption seems appropriate for the application in hand where B is much larger than W, and any input/output over W must be negligible in comparison with that over B. In this case we expect ϕ_F to be larger than ϕ_M. This is not necessarily true however. With $B = 0.5$ m and $W = 10$ m, the computation gave a value of $\phi_F = 0.064$, less than the ϕ_M value of 0.100. ϕ_F of course describes only the heat that is input over the width of the floor, and with so large a gap width, a substantial fraction of the total heat input must take place within the gap itself; by definition, there is no contribution of this kind to ϕ_M.

4. CORRECTION FOR FINITE LENGTH

As was shown in the last section, the term $\ln(2B/W+1)$ in equation (2) is part of an asymptotically exact solution for the case of an infinitely long floor. Macey extended it to a floor of finite L by examining the case of the square floor, arguing that "the greatest difference between an infinitely long floor and a rectangular one will obviously occur when the dimensions of the rectangle are the same in both directions, i.e., a square. If therefore the difference can be established between the heat loss from a square which is part of a very long floor (*on the one hand*—author's insertion) and one which stands by itself (*on the other*), and if this difference can be expressed as a ratio, (*then*) by interpolation the equivalent ratio can be estimated for rectangular floors of which the length differs from the width".

In the argument that followed, Macey made use of some previous computations by Keller for the loss of heat from a square surface to the surrounding ground, seemingly without mention of any wall thickness such as W. This led to a correction factor (CF) for a square of either 1.56 or 1.62, according to which of two starting

Table 3. Values of the correction factor CF

Ratio of sides of a reactangle $L:B$	CF	$\exp(B/2L)$
1:1	1.6	1.649
2:1	1.3	1.284
3:1	1.2	1.181
4:1	1.15	1.133
5:1	1.12	1.105
6:1	1.1	1.087
10:1	1.06	1.051
(∞:1	unity	1.000)

values was assumed. He chose a value of 1.6 and finally presented a table for values of CF (Table 3).

Macey did not suggest the exponential expression $\exp(B/2L)$ as a means of finding the correction factor, but it is clearly a reasonable empirical expression for his values and is given in the 1980 update for Section A3 of the CIBSE Guide [6]. It makes an inconsistency in this approach explicit however.

4.1. *The inconsistency of the generalization for finite length*

Although Macey's concern with a square floor is entirely proper since previous results were available, he cannot be correct in his contention that the 'greatest difference' between an ideal and a real surface must be when the real surface is square. A square is simply the crossover point between rectangles with L/B values greater than unity and less than unity, and no abrupt change should be evident in the U value as the ratio makes this transition. If equation (2) were a well-founded form for U value estimates, we should find the same value U if L and B reversed their identities: a floor of $100 \times 20\ \mathrm{m}^2$ say is the same viewed from either direction. Using the Guide expression however, the U values with reversed identities differ. With $W = 0.3$ m and $\lambda = 1.4\ \mathrm{W/m^2\,K}$ (the Guide's choice), using Anderson's formula

$$U = \frac{2 \times 1.4}{\pi} \cdot \frac{1}{20} \ln\left[\frac{2 \times 20 + 0.3}{0.3}\right] \cdot \exp\left[\frac{20}{2 \times 100}\right] = 0.2413\ \mathrm{W/m^2\,K} \quad (37)$$

and this agrees with the Guide value of $0.24\ \mathrm{W/m^2K}$. Viewed the other way round however,

$$U = \frac{2 \times 1.4}{\pi} \cdot \frac{1}{100} \ln\left[\frac{2 \times 100 + 0.3}{0.3}\right] \cdot \exp\left[\frac{100}{2 \times 20}\right] = 0.7062\ \mathrm{W/m^2\,K} \quad (38)$$

which is greater than the other value by a factor of nearly 3. Clearly L and B are not interchangeable.

Macey made it clear that the correction factor was to be applied only when L was the greater dimension and the CIBSE Guide too notes the restriction. U should not be found as in equation (38). The asymmetry in L and B however leads to another anomaly. It is obvious that in the case of a square floor, the decrease in U value by increasing either L or B must be the same. But this is not so as an example shows.

U can be conveniently expressed as

$$X = \frac{\pi U}{2\lambda} = \frac{1}{B} \ln\left[\frac{2B}{W} + 1\right] \exp\left[\frac{B}{2L}\right] = 0.6949 \quad (39)$$

when $L = 10$ m, $B = 10$ m and $W = 0.3$ m. If L is increased to 10.5 m,

$$X = 0.6785, \quad (40)$$

a decrease, as must be the case. Clearly, it makes no physical difference if instead B is increased to 10.5 m and then

$$X = 0.6863 \quad (41)$$

so

$$\delta X/\delta L = 0.0164 \quad \text{and} \quad \delta X/\delta B = 0.0086. \quad (42)$$

Thus $\delta U/\delta L$ is around double the value of $\delta U/\delta B$ although they should be the same. This cannot be: whatever merits the generalization to finite length may have in empirical terms, it is logically flawed.

4.2. *An alternative generalization*

(a) *Limitations to two-dimensional methods.* The expression for the field point temperature, $T_m = [q \ln(r_2/r_1)]/(2\pi\lambda)$, from which the floor heat loss, equations (23) or (24), can be eventually derived, is only true for an infinitely long line source. The expression itself is thus only true for an infinitely long floor; what might we do about a floor of finite length L?

A value for T_m, necessarily more complicated, can be written down for a finite length source and so the value of T_m due to a pair of line sources. The further arguments used earlier will not work here however. There are a number of reasons for this.

(i) When the cylinders are infinitely long, the floor surface and ground surface are quite separate from each other and so can have the different temperatures T_i and T_o. When the cylinders have finite length L, this is not so: the field point can be moved continuously from the 'floor' to the 'ground' and there can be no step change in temperature. Isothermal surfaces are no longer cylindrical in shape.

(ii) We might therefore attempt to close the system by addition of hot and cold line sources of length B at the ends of the L sources. All heat from the hot pair is transferred to the cold pair but now there are large temperature gradients in hot-cold corners and small gradients in the hot-hot and cold-cold corners and the model has an asymmetry not present in the parent situation.

(iii) If a transformation of an isotherm/heat-flow system to a heat-flow/isotherm system is attempted in three dimensions, a heat-flow line in the new system lies in the surface of a former isothermal surface but its actual direction is not further defined; similarly an isothermal surface in the new system contains the direction of the former heat flow line, but again is not further defined. It does not seem possible to use this transformation on the four line source configuration above, even if it were valid, to construct the system we really want: a rectangular floor at T_i separated on all four sides from an infinite area at T_o.

A rectangular floor is defined by two parameters, L and B. A circular one however is defined by its diameter alone and in this case it is possible to use the Macey approach to finding its heat loss, as is shown in Appendix 2.

(b) *A dimensionally-determined correction factor.* In the absence of a simple mechanistic argument, we will adopt a dimensional approach. We recognize that the expression for the heat loss Q must be symmetrical in L and B; the two basic dimensionless combinations of L and B are L/B, the 'shape' of the floor, and $\sqrt{(LB)}/W$, its 'size' in relation to the wall thickness. Consider however a combination of these quantities, $x = L \times B/(W(L+B))$: we note (i) that if L becomes large in relation to B, x tends to B/W and (ii) that if all dimensions for example double their values, x is unaltered. Thus if $2/x$ replaces $2W/B$ in equation (22), it provides an expression which is symmetrical in L and B, reproduces the earlier case when $L \gg B$, and leaves the heat loss per unit length of periphery unaltered when all dimensions are changed in proportion; this is strictly the case in equation (22) and one would expect to be true generally, at any rate approximately.

We expect therefore that a suitable generalization for the heat loss might be given by the expression,

$$Q = \lambda(T_i - T_o)\cdot(L+B)\frac{2}{\pi} \times\left[\ln\left[\frac{\sqrt{(1+2W(L+B)/LB)}+1}{\sqrt{(1+2W)(L+B)/LB)}-1}\right]+0.091\times\pi\right]. \quad (43)$$

For a square floor, small changes in L and B have of course an identical effect. We can check too that the expression is approximately correct for near-square floors. The heat loss from a length B in an effectively infinitely long floor is

$$Q = \lambda(T_i - T_o)\cdot B\cdot\frac{2}{\pi} \times\left[\ln\left[\frac{\sqrt{(1+2W/B)}+1}{\sqrt{(1+2W/B)}-1}\right]+0.091\times\pi\right]. \quad (44)$$

This is the heat loss too from a square floor positioned between vertical adiabatic surfaces so that heat can be lost normally into the x direction, into the areas A1 in Fig. 1, but not in the z direction into areas A2 and A3. If parts of the adiabatic volumes are removed so as to allow heat to flow additionally into the areas A2 but not A3, the heat loss must be approximately double that given equation (44). If now all adiabatic volumes are removed, so that heat can flow into all the surrounding ground, the heat loss will increase further and be given approximately by putting $L = B$ in equation (43):

$$Q = \lambda(T_i - T_o)\cdot 2B\frac{2}{\pi} \times\left[\ln\left[\frac{\sqrt{(1+4W/B)}+1}{\sqrt{(1+4W/B)}-1}\right]+0.091\times\pi\right] \quad (45)$$

which gives a value for Q more than twice the value of equation (44), as must be the case.

Equation (43) can therefore be taken as a generalization of Macey's expression, modifying it in three ways:

(i) It allows for the fact that in handling a line source/sink pair, the line sources are not concentric with the cylinders which may replace them. Thus with reversal of identities, the centre line of the wall is a little displaced from the point through which the isotherms tend to pass.
(ii) It allows approximately for the fact that the Macey approach tacitly assumes the presence of a semi-circular volume of perfectly insulating material beneath the wall location which partly blocks the heat loss.
(iii) It provides an allowance for a floor of finite length which is symmetrical in L and B.

It is not a rigorously arrived at expression and its value or otherwise has to be tested against the exact solution of Delsante *et al.* (Section 6). Since equation (36a) offers a more compact and accurate expression than equation (35) for the heat loss from a slab of infinite length, it too can be generalized to include floors of finite length, to be tested against the exact expression.

5. EFFECT OF INTERNAL CONVECTION AND RADIATION

Heat is transferred in a room by convection and radiation. For design purposes, these mechanisms are merged into a single film coefficient of value 8 or 9 W/m^{2}K or $R = 0.12$ m^{2}K/W in resistance form. R is included in finding the U value of a wall or roof, but is ignored in the Guide's values for ground floor losses. Does this matter?

For comparison purposes, it is convenient to define a ratio,

$$res = \frac{\text{resistance of the solid construction and outer film}}{\text{inside film resistance } (=0.12\ \text{m}^2\text{K/W})}. \quad (46)$$

For a single glazed window, *res* is about 0.46, for a solid 200 mm brick wall around 4.6 and for a wall where $U = 0.45$ W/m^{2}K it is 17.5. A value of *res* of about 10 or larger implies that the heat loss through the structure is not much affected by internal convection and radiation.

Using the simple Macey expression for a solid floor

$$res = \frac{\pi B}{R\cdot 2\lambda \ln(1+2B/W)} \simeq 9\frac{B}{\ln(1+6.7B)} \quad (47)$$

with $\lambda = 1.4$ W/m K and $W = 0.3$ m. Values of *res* are shown:

room width	B m	2	5	10	20
	res	6.7	13	21	37

Thus apart from quite narrow rooms, we can safely ignore inside film coefficients for overall loss calculations from the floor. This reflects the physical fact that the thermal resistance between points in the middle of a large floor and the exterior is large.

This result can be inferred from results published by Billington in 1951 [8] (p. 358). Using electrical analogue methods, he computed the U value of a floor 6.35 m in breadth, including inside and outside films, as 0.53 W/m^{2}K. When a layer of insulation of resistance 0.35 m^{2}K/W (some three times the film resistance) was laid across the floor, the U value was reduced by about 17% so one may argue that neglect of the film in the first place would lead to an underestimate of U by about 6%. (Billington went on to point out that insulation was better placed at the edge of the construction.)

Calculations based on U values however are often taken further to estimate the temperature distribution throughout a structure, possibly in connection with condensation, and in this case consideration must be given to conditions locally rather than globally: the local resistances must be used.

At a point x from the centreline of a floor,

$$res(x) = \frac{\pi[\frac{1}{4}(B+W)^2 - x^2)]}{R \cdot \lambda(B+W)} \tag{48}$$

and at the edge, where $x = \frac{1}{2}B$,

$$res(\tfrac{1}{2}B) = \frac{\pi[\frac{1}{2}W(B+\frac{1}{2}W)]}{R \cdot \lambda(B+W)} \quad \text{or} \quad \frac{\frac{1}{2}\pi W}{R \cdot \lambda} \text{ nearly} \tag{49}$$

The last result follows directly from the Macey model: the path length round the adjacent semicylinder of perfect insulating material is $\frac{1}{2}\pi W$ and so the resistance is $\frac{1}{2}\pi W/\lambda$. This value is of course too large: the value should be nearer W/λ. Taking it at its face value however, the resistance ratio *res* is 2.8, a value which would not be ignored in wall U value calculations.

Since it is unrealistic to suppose that the floor temperature must remain at the strictly uniform temperature T_i over its whole area, it follows that its distribution $T(x)$ can be found approximately as

$$\frac{T(x) - T_o}{T_g - T_o} = \frac{R(\text{floor})}{R(\text{floor}) + R(\text{film})} = \frac{res(x)}{res(x)+1} \tag{50}$$

where T_g is the room global temperature.

The temperature falls away toward the edge, somewhat more so than is estimated by this expression. It is of course further influenced by the actual wall construction that occupies the width W.

6. DISCUSSION

The total heat loss from a floor of area $L \times B$ surrounded by a wall of thickness W through heat conducting material of conductivity λ can be written as

$$Q = \lambda(T_i - T_o) \cdot G \tag{1}$$

where G is a geometrical function of L, B and W.

Delsante *et al.* have provided an exact relation for G, ([4], eq 25). The solution rests on the assumption of a uniform temperature gradient at floor level over the area below the wall. This is a reasonable assumption but one which cannot in fact be realized. The wall and floor form a coupled system. With the assumption of a linear gradient, there must be heat flow from the wall into the floor on the room side of the wall centre line, and a flow from the floor into the wall on the other side. Thus the floor and ground bring about a greater heat loss than is associated with the floor area $L \times B$ strictly interpreted. One might have assumed instead that the floor surface at ground level under the wall was an adiabatic surface (in which case, the temperature gradient varies with position). When W is small in relation to L or B either assumption is likely to lead to much the same results but as pointed out earlier (end of Section 3), this is not the case when $B < W$.

Five approximate forms for G commend themselves.

(i) A rearrangement of Macey's original expression as given in the CIBSE Guide has

$$G = \frac{2}{\pi} \cdot L \cdot \ln\left[\frac{2B}{W} + 1\right] \cdot \exp\left[\frac{B}{2L}\right]. \tag{2}$$

(ii) In the modified version presented here as equation (43),

$$G = \frac{2}{\pi} \cdot (L+B) \cdot \left[\ln\left[\frac{\sqrt{(1+2/x)}+1}{\sqrt{(1+2/x)}-1}\right] + 0.091 \times \pi\right]$$

$$\text{where} \quad x = \frac{L \times B}{W(L+B)}. \tag{51}$$

(iii) In the version based on the exact two dimensional form from [4] or [9], equation (36a),

$$G = \frac{2}{\pi} \cdot (L+B) \cdot \ln\left[(1+x)\left(1+\frac{1}{x}\right)^x\right]$$

$$\text{where} \quad x = \frac{L \times B}{W(L+B)}. \tag{52}$$

(iv) In the limiting or approximate value of Delsante, Stokes and Walsh based on an expansion of each factor in their exact solution to include terms in W^{-1} and W^0, which is valid when L and B are large compared with W:

$$G = \frac{2}{\pi} \cdot \left[L \ln\left[\frac{2L}{W}\right] + B \ln\left[\frac{2B}{W}\right] + 2(L^2+B^2)^{1/2} - L - B \right.$$
$$\left. - B \ln \frac{(L^2+B^2)^{1/2}+B}{L} - L \ln \frac{(L^2+B^2)^{1/2}+L}{B}\right]. \tag{3}$$

(The last five terms do not include W.)

(v) In Delsante *et al*'s solution, further expanded to include terms in W^{+1}:

$$G = \frac{2}{\pi} \cdot \left[L \ln\left[\frac{2L}{W}\right] + B \ln\left[\frac{2B}{W}\right] + 2(L^2+B^2)^{1/2} \right.$$
$$- L - B - B \ln \frac{(L^2+B^2)^{1/2}+B}{L} - L \ln \frac{(L^2+B^2)^{1/2}+L}{B}$$
$$+ \frac{W}{2}\left[\ln \frac{4L^2B^2}{W^2 \cdot ((L^2+B^2)^{1/2}+B)((L^2+B^2)^{1/2}+L)}\right.$$
$$\left.\left. + \frac{(L+B)(L^2+B^2)^{1/2}}{LB} + 2\sqrt{2} - 1 + 2 \cdot \ln(\sqrt{2}-1)\right]\right]$$

(53)

These forms can be checked against the exact (17 term) solution G(exact) of Delsante *et al.* A comparison was made, structured after the table of U values in the CIBSE Guide, in which $W = 0.3$ m. The last five columns of Table 4 provide values of (i) G(Macey)/G(exact), (ii) G(Macey modified)/G(exact), (iii) G(exact 2D)/G(exact), (iv) G(Delsante, approximate)/G(exact) and (v) (Delsante, further approximation)/G(exact). The table also lists the U values based on the exact solution and a value of $\lambda = 1.4$ W/m^{2}K (the Guide's choice), computed as $U = 1.4G/LB$. (They are a little larger than those listed in the Guide, which are of course based on Macey's expression.)

It will be apparent that heat losses calculated using the Macey expression must be some 10% low; this corresponds to the 'presence' of perfectly insulating material just where the heat flux is largest. Losses calculated by the modified Macey expression appear to be mostly within 2% of their exact value and the form based on the exact 2D expression, equation (52), reduces the deviation for the most part to less than 1%. This is discussed in Appendix 3. When $x \gg 1$, the floor U value found from equation (52) becomes identical with that recently proposed by Anderson [9], see his equation (23). Delsante *et al*'s approximate form (iv) varies from being near exact for large floors to be being a few percent low for small floors. The further approximation (v) improves the accuracy as expected.

In this problem, the exact solution for G does not help to indicate any nondimensional groupings which may serve to parameterize numerical values for G. It is made up of a series of terms in L, B and W, each of which has the units of length but there seems no point in trying to draft it in terms of any dimensionless variables. The

Table 4. Comparison of expressions for heat losses from solid uninsulated floors

Length L m	Breadth B m	U value W/m^{2}K	G(Macey)/G(exact)	G(Macey, modified)/G(exact)	G(2D exact)/G(exact)	G(Delsante approx)/G(exact)	Delsante further approx/G(exact)
2000	2000	0.008	0.867	1.007	1.009	1.000	1.000
2000	100	0.063	0.941	0.999	1.002	1.000	1.000
2000	60	0.096	0.943	0.998	1.001	0.999	1.000
2000	40	0.133	0.943	0.998	1.001	0.999	1.000
2000	20	0.234	0.939	0.998	1.000	0.998	1.000
2000	10	0.404	0.931	0.999	1.000	0.997	1.000
2000	6	0.598	0.923	1.001	1.000	0.994	1.000
2000	4	0.809	0.915	1.004	1.000	0.990	1.000
2000	2	1.324	0.897	1.015	1.000	0.976	1.001
100	100	0.108	0.887	1.009	1.012	0.998	1.000
100	60	0.137	0.877	1.009	1.012	0.998	1.000
100	40	0.172	0.886	1.007	1.010	0.997	1.000
100	20	0.267	0.904	1.004	1.007	0.996	1.000
100	10	0.432	0.913	1.003	1.004	0.994	1.000
100	6	0.623	0.913	1.003	1.002	0.991	1.000
100	4	0.831	0.909	1.005	1.001	0.987	1.000
100	2	1.341	0.894	1.015	1.000	0.973	1.001
60	60	0.165	0.892	1.010	1.013	0.997	1.000
60	40	0.198	0.879	1.009	1.012	0.996	1.000
60	20	0.290	0.890	1.007	1.009	0.995	1.000
60	10	0.452	0.904	1.005	1.006	0.993	1.000
60	6	0.640	0.907	1.005	1.003	0.990	1.000
60	4	0.846	0.904	1.006	1.002	0.986	1.000
60	2	1.353	0.892	1.015	1.000	0.972	1.001
40	40	0.229	0.896	1.010	1.013	0.996	1.000
40	20	0.318	0.883	1.009	1.011	0.994	1.000
40	10	0.476	0.895	1.007	1.007	0.991	1.000
40	6	0.661	0.900	1.006	1.004	0.988	1.000
40	4	0.864	0.900	1.007	1.003	0.984	1.000
40	2	1.368	0.890	1.016	1.000	0.970	1.001
20	20	0.398	0.904	1.011	1.012	0.991	1.000
20	10	0.546	0.884	1.010	1.010	0.987	1.000
20	6	0.722	0.887	1.009	1.007	0.983	1.000
20	4	0.919	0.889	1.010	1.004	0.979	1.000
20	2	1.412	0.884	1.017	1.000	0.964	1.001
10	10	0.678	0.913	1.012	1.010	0.981	1.000
10	6	0.841	0.886	0.012	1.008	0.975	1.000
10	4	1.026	0.881	0.013	1.005	0.970	1.000
10	2	1.498	0.875	0.019	1.000	0.954	1.000
6	6	0.989	0.920	1.014	1.007	0.968	1.000
6	4	1.161	0.889	1.015	1.004	0.964	1.000
6	2	1.609	0.871	1.021	0.998	0.942	1.000
4	4	1.320	0.924	1.017	1.002	0.951	1.000
4	2	1.742	0.874	1.022	0.996	0.929	0.999
2	2	2.108	0.928	1.028	0.990	0.898	0.998

approximate solution, equation (3), after re-arrangement, can be expressed in terms of $\sqrt{(LB)}/W$ and L/B. The $LB/W(L+B)$ grouping, which in fact accounts for nearly all the variation in G in the range of values of L, B and W of practical interest, only makes itself evident in numerical evaluation.

7. CONCLUSIONS

Which is the most suitable expression for design purposes? A number of factors have to be considered in relation to floor U values.

(i) Apart from the effect of the inside film resistance, U is proportional to the ground conductivity, and this is reported to vary from 0.7 to 2.1 W/m K. Knowledge of actual ground conditions will narrow this uncertainty, but a band of uncertainty must remain.

(ii) U values for large floors are inherently small: some of those listed in the Guide are less than the value of 0.45 W/m^{2}K required for walls in the current U.K. regulations [10]. The effective U values could well be less than these because deep ground temperature is higher than average ambient temperatures in the heating season.

(iii) According to [10] Diagram 2, insulation is not required for a floor greater than about 15×15 m^2. Use of insulation near the floor perimeter will substantially reduce the heat loss.

These considerations suggest that U values for uninsulated floors neither can nor necessarily need to be known accurately. Since there is uncertainty in λ, the original Macey expression (i) could be regarded as sufficient, but the defects discussed in this article dent its credibility.

In the modified form (ii), the correction for the insulated semi-cylinder is rationally based, but that for length is based on dimensional rather than upon mechanistic arguments, which would have been preferable. (iii) is asymptotically exact, but modified in the same way as (ii).

The approximation advanced by Delsante, Stokes and Walsh (iv) is based on an exact analysis and holds when W is small compared with L and B. On rational grounds it would seem an attractive form. However, (iv) provides less accuracy than (iii) for small floors with large U values, where accuracy is in fact most needed; (iv) is also less compact than (iii). The accuracy given by (v) is excellent but compared with (iii) it is a cumbersome expression.

We may conclude therefore that as a simple working formula, (iii)—equation (52)—seems the most suited to computing heat losses from solid uninsulated ground floors. It leads to an expression for the floor transmittance,

$$U = \frac{\lambda}{W}\cdot\frac{2}{\pi x}\ln\left[(1+x)\left(1+\frac{1}{x}\right)^{x}\right]. \quad (54)$$

With $L = B = 20$ m and $W = 0.3$ m for example, $x = 33.3$ and

$$U = \frac{\lambda}{W}\cdot 0.086. \quad (55)$$

The corresponding expression for the transmittance of a vertical wall (without film resistances) is simply

$$U' = \frac{\lambda}{W}. \quad (56)$$

Equation (54) thus shows the form of the floor loss in the same terms, λ and W, as the wall loss; the effect of ground geometry is shown non-dimensionally by x.

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APPENDIX 1
THE SYSTEMS OF ORTHOGONAL CIRCLES

Macey's approach to the floor heat loss is only possible because the isothermal lines in the ground happen to be circular. It is useful to demonstrate that this is in fact so, and that the heat flow lines form an orthogonal circular set. It is convenient to go back to the original model of parallel hot and cold line sources. See Fig. 5.

The centres of the two sources are notated as M_1 and M_2. r_1 denotes the distance from M_1 to some field point P. The temperature at P is $T_m = q\ln(r_2/r_1)/(2\pi\lambda)$ so that if P moves in such a way that r_2/r_1 (equal to α say) is constant, P defines an isothermal line. We denote by $2d$ the distance between the line sources. Then

$$\frac{r_2^2}{r_1^2} = \alpha^2 = \frac{(x+d)^2+y^2}{(x-d)^2+y^2}. \quad (A1)$$

This can be written in the form,

$$\left[x - d\frac{\alpha^2+1}{\alpha^2-1}\right]^2 + y^2 = \left[\frac{2\alpha d}{\alpha^2-1}\right]^2. \quad (A2)$$

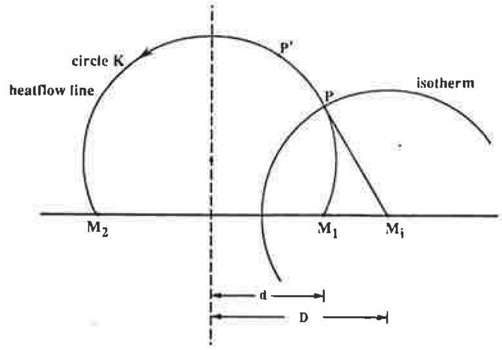

Fig. 5. To illustrate that the heat flow lines and isotherms associated with conduction between a hot-cold pair of line sources form two orthogonal systems of circles.

However, the equation of a circle of radius c and centre (the point M_i) a distance D along the x axis is

$$(x-D)^2+y^2 = c^2. \tag{A3}$$

So by identification of these equations, the isotherm has the form of a circle whose centre is a distance D from the central plane, and radius c, given by

$$D = d\frac{\alpha^2+1}{\alpha^2-1} \quad \text{and} \quad c = \frac{2\alpha d}{\alpha^2-1}$$

(A4.1) and (A4.2)

so $D^2-d^2 = c^2$ as given earlier (equation 13).

(If the field point P is on the horizontal axis (so that $r_2+r_1 = 2d$),

$$\frac{r_2}{r_1} = \alpha = \frac{1+\sqrt{(1-c^2/D^2)}}{c/D}.$$

The relations between the geometrical construction and the building are that $2c \equiv W$ and that $D-c \equiv B/2$. So $c/D = W/(B+W)$ and is small compared to 1 in a building context. Thus $\alpha \gg 1$. Now $D-d = 2d/(\alpha^2-1)$ and tends to zero when $\alpha \gg 1$. In building applications with reversed identities, D denotes the location of the centre line of the wall, and d, the location of the virtual point through which the isotherms tend to pass. These practically coincide. Macey tacitly assumed this in his analysis. The exact expression for the heat loss, equation (22), reduces to the Macey form, equations (23) or (24), if $W/(B+W) \ll 1$.)

The heat flow lines for the line sources model are orthogonal to the isotherms and one might guess that they too would be circular. To check that this is indeed so, we note that

$$M_iM_1 \cdot M_iM_2 = (D-d)(D+d) = D^2-c^2 = d^2\left[\frac{\alpha^2+1}{\alpha^2-1}\right]^2 - d^2$$

$$= \frac{4\alpha^2d^2}{(\alpha^2-1)^2} = c^2 = M_iP^2 \tag{A5}$$

where P is some point on the isotherm centred at M_i.

Now the relation $M_iM_1 \cdot M_iM_2 = M_iP^2$ is true for the circle (K say) that passes through M_1, M_2 and P, and M_iP ($=c$) is tangent to circle K at P. Now c is of course the radius of the isotherm, and another point P' can similarly be located on circle K, so that $M_i'P'$ is the radius of another isotherm. Thus a tangent at any point on circle K forms the radius of an isotherm. But the radius M_iP of an isotherm necessarily indicates the direction of heat flow at P. Thus the direction of the heat flow lines is too circular.

The heat flow lines leave the line sources radially and uniformly and so a system of heat flow tubes carrying equal heat flows can readily be set up. (If a heat flow line makes an angle θ below the horizontal at M_1, the circle centre is at $y = +d/\tan\theta$.) When the line sources are replaced by cylinders which coincide with an isothermal surface of the parent line source system, the heat flow lines originate at the cylinder surfaces and leave normally. They are not however uniformly spaced around the cylinders: they tend to cluster in the space between the cylinders and spread sparsely into the space away from the cylinders.

Returning to the Macey problem, it is clear that the positions of the isotherms in the ground below and outside the room can be found exactly as the flow lines above: they are circles with centres on the vertical axis and for a fairly large room, the isotherm midway between T_i and T_o will be the circle centred on $x = 0$, $y = 0$ and radius $\frac{1}{2}(B+W)$.

We can also locate flow lines within the ground. We want to be able to draw the flow line in the right hand half of the flow field such that a given fraction f of the total heat flow to the right of the centre line is contained between this flow line and the centre line. Suppose the flow line cuts the surface a distance x' from the centre line. Integration of equation (19) from $x = 0$ to $x = \frac{1}{2}B$ gives us the semi heat flow from the room to exterior, and integration from $x = 0$ to $x = x'$ gives us a fraction f of this semi flow. It is readily found that x' and f are related as

$$\frac{\sqrt{(1+2W/B)}+2x'/B}{\sqrt{(1+2W/B)}-2x'/B} = \left[\frac{\sqrt{(1+2W/B)}+1}{\sqrt{(1+2W/B)}-1}\right]^f. \tag{A6}$$

Now for any point P on the circular flow line,

$$\frac{r_2}{r_1} = \alpha, \quad \text{a constant value} \tag{A7}$$

and when P is on the room floor,

$$\alpha = \frac{r_2}{r_1} = \frac{d+x'}{d-x'} = \frac{\sqrt{(1+2W/B)}+2x'/B}{\sqrt{(1+2W/B)}-2x'/B}. \tag{A8}$$

Thus α is related to the fractional heat loss f as

$$\alpha = \left[\frac{\sqrt{(1+2W/B)}+1}{\sqrt{(1+2W/B)}-1}\right]^f. \tag{A9}$$

Then the position of the centre of the heat flow line, distance D from the centre line, and its radius c are given by the equations (A4.1) and (A4.2) above. It is easily checked that if $B \gg W$ and also that $f = 1$, then $D = \frac{1}{2}(B+W)$ and $c = \frac{1}{2}W$, as must of course be the case.

If the heat flow lines are located such that, for example $f = 0$ (the vertical centre line), $f = \frac{1}{4}$, $f = \frac{1}{2}$, $f = \frac{3}{4}$ and $f = 1$ (coincident with the insulating semi-cylinder), it is found that the lines tend to cluster near the edge of the floor and this is consistent with the relatively large loss of heat from a slab near its edge.

If these lines are taken to delimit the heat tubes, all tubes defined by equal increments δf of f necessarily have the same resistance—$R(\delta f)$ say—in the ground. The total resistance between the room global temperature and the exterior must however include the inside film resistance and tubes terminating near the room centre have a larger cross section area and so a lower film resistance than tubes terminating near the edge. This provides an approach to estimating the variation in floor temperature. The argument is not exact of course: the flow lines have been located on the assumption that the floor is an isothermal surface. If we go over to assuming that it is the *total* resistance, film and ground, between inside and outside that is to be the same in each tube, the pattern of flow lines and isotherms in the ground must become a little distorted from its simple circle-structure.

APPENDIX 2
HEAT LOSS FROM A CIRCULAR FLOOR

The heat loss from a circular floor, diameter B' at T_i, surrounded by a wall of width W, to an external surface at T_o can be found in manner similar to that for a long rectangular floor. Suppose that the figure in Fig. 3 is rotated about its vertical axis: the circular isotherms necessarily trace out spherical surfaces, and the circular heat flow lines trace out toroids everywhere orthogonal to the isothermal surfaces. The new system therefore describes the heat flow between the horizontal isothermal circular surface (diameter B') within the 'wall' toroid (width W) and the horizontal surface outside the toroid.

The argument for finding the temperature gradient dT/dy remains the same as previously. The elementary area at which it acts however is now $2\pi x \cdot dx$ (instead of $L \cdot dx$). The heat flow is then found by integration from $x = 0$ to $x = \frac{1}{2}B'$. As before, the fictitious distance d is related to the radius D of the semi-circular section, circular form perfect insulator as

$$d^2 = D^2 - (W/2)^2. \quad \text{(A10)}$$

and

$$2D = B' + W. \quad \text{(A11)}$$

Then

$$Q = \lambda(T_i - T_o)B'\sqrt{(1+2W/B')}\ln\left[1+\frac{B'}{2W}\right]. \quad \text{(A12)}$$

To this should be added a correction for the blockage imposed by the insulating toroid.

APPENDIX 3
THE NON-DIMENSIONALIZED HEAT LOSS H

The heat loss Q (watts) from the floor area $L \times B$ is

$$Q = \lambda(T_i - T_o) \cdot G. \quad \text{(1)}$$

The non-dimensionalized form of Q can be written

$$H = \frac{Q}{\lambda(T_i - T_o) \cdot 2(L+B)} = \frac{G}{2(L+B)}. \quad \text{(A13)}$$

(The factor of 2 is included since the total periphery is $2(L+B)$.) In the approximation based on the exact two dimensional form (equation (52)),

$$H = \frac{1}{\pi} \cdot \ln\left[(1+x)\left(1+\frac{1}{x}\right)^x\right]. \quad \text{(A14)}$$

In the exact form for heat loss due to Delsante *et al*, H has a further parameter, conveniently chosen as L/B. When L/B is very large or very small, the Delsante *et al* expression reduces to the two dimensional form of (A14) and will differ most from it for a square floor ($L = B$). Table 5 lists values of H for L/B = infinity and $L/B = 1$.

$H(2D)$ and H(square) differ by several percent for small values of x, but these are not of interest in a building context. For realistic values, they differ by around 1%. It was mentioned earlier that the variable x takes account of both the size and the shape of the floor/wall system: evidently the heat loss depends only weakly on further inclusion of the shape of the floor.

Table 5. Values of the non-dimensionalized heat loss H from a solid floor, $L \times B$, for L/B = infinity (the two dimensional case), and $L/B = 1$ as a function of $x = L \times B/(W(L+B))$

x	$L/B = \infty$	$L/B = 1$
0.1	0.107	0.126
0.2	0.172	0.196
0.3	0.224	0.250
0.4	0.267	0.294
0.5	0.304	0.331
0.7	0.367	0.393
1.0	0.441	0.465
1.5	0.536	0.555
2.0	0.608	0.623
3.0	0.716	0.725
4.0	0.796	0.801
5.0	0.861	0.862
7.0	0.959	0.957
10.0	1.067	1.060
15.0	1.191	1.179
20.0	1.280	1.266