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The Effect of Water Table Depth on Steady-State Heat Transfer Through a Slab-on-Ground Floor

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The effect of water table depth and temperature on the total heat flux through a slab-on-ground floor has not been previously analysed explicitly. In this paper the two-dimensional problem is solved using a conformal transformation. An expression for the total flux from the floor is obtained and compared with the total flux without a water table. Results are presented graphically for a water table temperature equal to the outdoor temperature, for realistic ranges of two governing parameters: the ratio of building width to water table depth, and the ratio of wall thickness to building width. For any water table depth greater than the building width, the flux is found to be within 10% of the flux without a water table.

INTRODUCTION

THE PROBLEM of calculating heat losses from slab-on-ground floors has been tackled in a variety of ways, ranging from analytical treatments using simplified boundary conditions (e.g. [1–3]), semi-analytical treatments with possibly more realistic boundary conditions (e.g. [4–6]), through to fully numerical treatments, which in principle can deal with complex boundary conditions (e.g. [7]). The boundary condition at some distance below the slab is of particular interest in this paper. In the analytical solutions of Delsante [2] and Anderson [3], for example, the temperature was only specified at the surface, which determined the temperature field at any depth. Their results thus excluded the possibility of a water table at some finite depth imposing a temperature or other boundary condition there. Other treatments, such as that of Krarti *et al.* [5], assumed a water table with an associated temperature boundary condition, but did not examine the effect of the water table position and temperature on total heat losses from a floor slab.

In this paper, the two-dimensional steady-state problem with a finite water table depth will be solved using conformal transformation, and the results compared with those obtained in the absence of a water table.

THE MODEL AND BOUNDARY CONDITIONS

The geometry of the two-dimensional steady-state problem is shown in Fig. 1. The ground is represented by the region $y \geqslant 0$; a water table at $y = L$ at temperature T_w represents an isothermal line at this depth, and is assumed to impose a temperature boundary condition there. This representation of the boundary condition imposed by the water table is the same as was used in [5–7]; an alternative and possibly more realistic representation would be to assume that the ground thermal conductivity changes at $y = L$ (from a 'dry' to a 'wet' value); however this would then preclude the use of the simple technique used here to obtain solutions. Another representation was used by Hagentoft [8], who assumed that the thermal conductivity was the same above and below the water table, and calculated the heat removed by a given flow rate of water.

The thermal conductivity of the ground above the water table is assumed to be uniform, and equal to that of the floor slab (a reasonable approximation for concrete). The building floor extends from $-a$ to a, with walls of thickness d placed at a and $-a$. To simplify the geometry, the floor is assumed to be flush with the ground surface. A surface temperature boundary condition will be

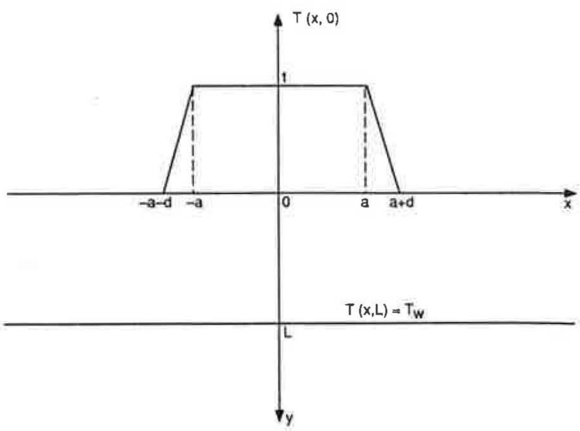

Fig. 1. Boundary conditions and geometry of the model for a slab-on-ground floor with a water table. The ground is the two-dimensional region $y \geqslant 0$, with a water table at $y = L$ at temperature T_w. The floor extends from $-a$ to a, and walls of thickness d are placed at $-a$ and a. The surface temperature boundary condition $T(x, 0)$ is also shown.

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BAE 28:3-J

assumed to allow analytical solutions to be obtained. A more realistic boundary condition would be the temperature of the air above the slab (see Delsante [2] for a treatment of this case in the absence of a water table). However, for the purposes of comparing heat losses with and without a water table, the surface boundary condition should be adequate. We may assume that the temperatures have been scaled so that the indoor surface temperature is one, and the outdoor surface temperature is zero. Furthermore, we assume that the surface temperature changes linearly from its indoor to its outdoor value over the wall thickness. This is the simplest way of ensuring a continuous change from indoors to outdoors; a discontinuous change in the surface temperature boundary condition at the slab edge leads to intractable divergences in the heat flux there (although it does not if the boundary condition is the temperature of the air above the surface), and is therefore undesirable.

For ease of transformation, let us rescale lengths by π/L, so that the water table is at $y = \pi$, and let $\alpha = a\pi/L$ and $\delta = d\pi/L$ denote the rescaled building half-width and wall thickness respectively. Figure 2 shows the rescaled geometry, with points of interest indicated by the letters A to J. In the steady state the temperature field $T(x, y)$ satisfies

$$\nabla^2 T(x, y) = 0;$$

$$T(x,0) = \begin{cases} 1, & |x| \leqslant \alpha, \\ 0, & |x| \geqslant \alpha+\delta, \\ (\alpha+\delta-|x|)/\delta, & \alpha \leqslant |x| \leqslant \alpha+\delta; \end{cases}$$

$$T(x, \pi) = T_w.$$

Consider now a conformal mapping from the complex plane $z = x+iy$ to the complex plane $w = u+iv$, given by

$$w = \exp(z).$$

Applying this transformation to Fig. 2, where (x, y) plane is taken to represent the complex plane, yields Fig. 3, in which it can be seen that the water table at $z = x+i\pi$ has been transformed to the region $u < 0$, $v = 0$. The transformation of the points of interest A to J is also indicated.

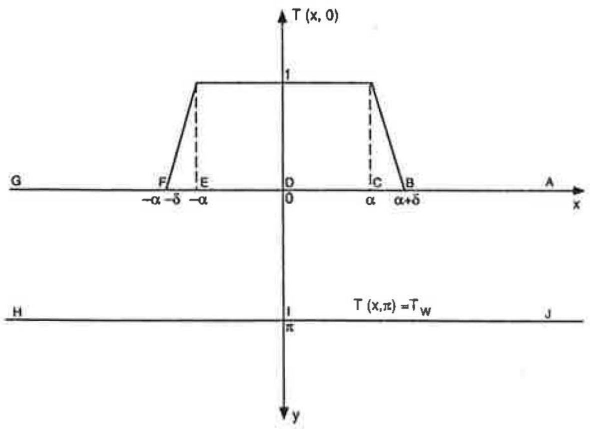

Fig. 2. As for Fig. 1, with dimensions rescaled by π/L, so that the water table is now at $y = \pi$, with $\alpha = a\pi/L$ and $\delta = d\pi L$. Points of interest are indicated by the letters A to J.

In the w-plane, the temperature field $T(u, v)$ is still harmonic and therefore satisfies

$$\nabla^2 T(u, v) = 0;$$

$$T(u,0) = \begin{cases} T_w, & u < 0, \\ 0, & 0 \leqslant u \leqslant \exp(-\alpha-\delta), \\ (\alpha+\delta+\ln(u))/\delta, & \exp(-\alpha-\delta) \leqslant u \leqslant \exp(-\alpha), \\ 1, & \exp(-\alpha) \leqslant u \leqslant \exp(\alpha), \\ (\alpha+\delta-\ln(u))/\delta, & \exp(\alpha) \leqslant u \leqslant \exp(\alpha+\delta), \\ 0, & u \geqslant \exp(\alpha+\delta). \end{cases} \qquad (1)$$

SOLUTIONS

Equation (1) is a problem for the semi-infinite solid with a surface temperature boundary condition only, the solution of which has been given by Anderson [3] as

$$T(u, v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2v}{v^2+(u-t)^2} T(t, 0)\,dt. \qquad (2)$$

Before proceeding with the calculation of the total flux from the slab, we note that (1) and (2) may be used to obtain the temperature field for a step change in the surface temperature (i.e. for $\delta = 0$). The result is

$$T(x, y) = \frac{T_w y}{\pi} + \frac{1}{\pi}\left[\arctan\left\{\frac{e^x \cos y - e^{-\alpha}}{e^x \sin y}\right\} - \arctan\left\{\frac{e^x \cos y - e^{\alpha}}{e^x \sin y}\right\}\right], \qquad (3)$$

valid for all x and y (although care must be taken at $y = 0$). This expression may be compared with that obtained by Krarti *et al.* [5] for identical boundary conditions. They give the result in two parts, one valid for $x < -\alpha$, the other for $-\alpha \leqslant x \leqslant 0$ the temperatures for $x > 0$ are the same because of symmetry). For $x < -\alpha$ their result is identical to (3), but for $-\alpha \leqslant x \leqslant 0$ it is not; furthermore their two expressions are not continuous at $x = -\alpha$, although their graphed isotherms appear to be. Within the accuracy limitations of the graphs, (3) appears to reproduce their isotherms very well.

The total heat flux from the slab, Φ, is given by

$$\Phi = \int_{-\alpha}^{\alpha} -k\left.\frac{\partial T}{\partial y}\right|_{y=0} dx,$$

or

$$\Phi = \int_{\exp(-\alpha)}^{\exp(\alpha)} -k\left.\frac{\partial T}{\partial v}\right|_{v=0} du, \qquad (4)$$

where k is the thermal conductivity of the ground. From (1) and (2), we find

$$\left.\frac{\partial T}{\partial v}\right|_{v=0} = \frac{T_w}{\pi u} + \frac{1}{\pi u \delta} \times [\ln|u-\exp(-\alpha)| - \ln|u-\exp(-\alpha-\delta)| - \ln|u-\exp(\alpha+\delta)| + \ln|u-\exp(\alpha)|]. \qquad (5)$$

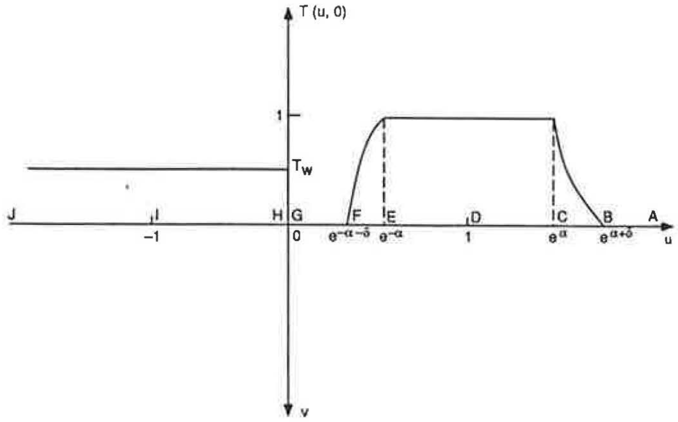

Fig. 3. Effect of applying the conformal transformation $w = u+iv = \exp(z)$ to the complex plane $z = x+iy$ of Fig. 2, showing the location of the transformed points A to J. The water table at $y = \pi$ has been transformed to the region $u < 0$, $v = 0$. The linear change in the surface temperature of Fig. 2 has been transformed to a logarithmic change.

Subsituting (5) into (4) gives, after some manipulation,

$$\Phi = \frac{2\alpha k}{\pi}(1-T_w) - \frac{2k}{\pi\delta}\left[\int_0^{1-\exp(-2\alpha)} \frac{\ln t}{1-t}\,dt - \int_{1-\exp(-\delta)}^{1-\exp(-2\alpha-\delta)} \frac{\ln t}{1-t}\,dt\right]. \quad (6)$$

An explicit expression for the integrals in (6) is not available. However, we can use the expansion

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

to obtain

$$\int \frac{\ln t}{1-t}\,dt = \ln t \sum_{n=1}^{\infty} \frac{t^n}{n} - \sum_{n=1}^{\infty} \frac{t^n}{n^2},$$

or

$$\int \frac{\ln t}{1-t}\,dt = -\ln t \ln(1-t) - \sum_{n=1}^{\infty} \frac{t^n}{n^2}. \quad (7)$$

Applying (7) to (6) gives

$$\Phi = \frac{2\alpha k}{\pi}(1-T_w) + \frac{2k}{\pi}\left[\ln\left(\frac{1-\exp(-2\alpha-\delta)}{1-\exp(-\delta)}\right) + \frac{2\alpha}{\delta}\ln\left(\frac{1-\exp(-2\alpha-\delta)}{1-\exp(-2\alpha)}\right)\right] + \frac{2k}{\pi\delta}\sum_{n=1}^{\infty}\frac{1}{n^2} \times [(1-\exp(-2\alpha))^n + (1-\exp(-\delta))^n - (1-\exp(-2\alpha-\delta))^n]. \quad (8)$$

This expression can be checked in the limit of infinite water table depth, $L \to \infty$. In this limit, $\alpha \to 0$, $\delta \to 0$, and α/δ (i.e. a/d) remains constant, since lengths have been scaled by π/L. Hence the total flux in this limit, Φ_∞, obtained by expanding the exponentials in (8), is given by

$$\Phi_\infty = \frac{2k}{\pi}\left[\ln\left(\frac{2\alpha+\delta}{\delta}\right) + \frac{2\alpha}{\delta}\ln\left(\frac{2\alpha+\delta}{2\alpha}\right)\right]. \quad (9)$$

This is the result obtained by Delsante *et al.* [1] and Anderson [3].

The infinite series converges rapidly enough to allow (8) to be readily used for practical calculations: typically, 50 terms give an accuracy of better than 0.2% for parameter ranges typical of buildings (see below).

RESULTS AND DISCUSSION

In assessing the effect of water table depth on the heat flux, the parameters of interest that arise are the ratio of building width to water table depth, $2a/L$, and the ratio of wall thickness to building width, $d/2a$. Let $\gamma = 2a/L$ and $\beta = d/2a$; then (8) can be written as

$$\Phi = \gamma k(1-T_w) + \frac{2k}{\pi}\left[\ln\left(\frac{1-\exp(-\pi\gamma(1+\beta))}{1-\exp(-\pi\gamma\beta)}\right) + \frac{1}{\beta}\ln\left(\frac{1-\exp(-\pi\gamma(1+\beta))}{1-\exp(-\pi\gamma)}\right)\right] + \frac{2k}{\pi^2\gamma\beta}\sum_{n=1}^{\infty}\frac{1}{n^2} \times [(1-\exp(-\pi\gamma))^n + (1-\exp(-\pi\gamma\beta))^n - (1-\exp(-\pi\gamma(1+\beta)))^n]. \quad (10)$$

Equation (10) gives the total flux from the slab for a continuous surface temperature boundary condition as a function of water table depth and temperature. Krarti *et al.* [5] give a graph showing the effect of water table depth on the heat flux distribution along an uninsulated slab (i.e. the flux as a function of x at $y = 0$) for a discontinuous surface temperature condition. As would be expected from their temperature boundary condition, the flux diverges at the slab edge. More importantly, it can be seen from (10) that the total flux from the slab will also diverge with this boundary condition. Such divergences confuse the interpretation of the results, and illustrate the importance of imposing a continuous surface temperature condition (or alternatively locating the step change in surface temperature in the centre of the wall, say, and only integrating up to the wall; for example see [3]).

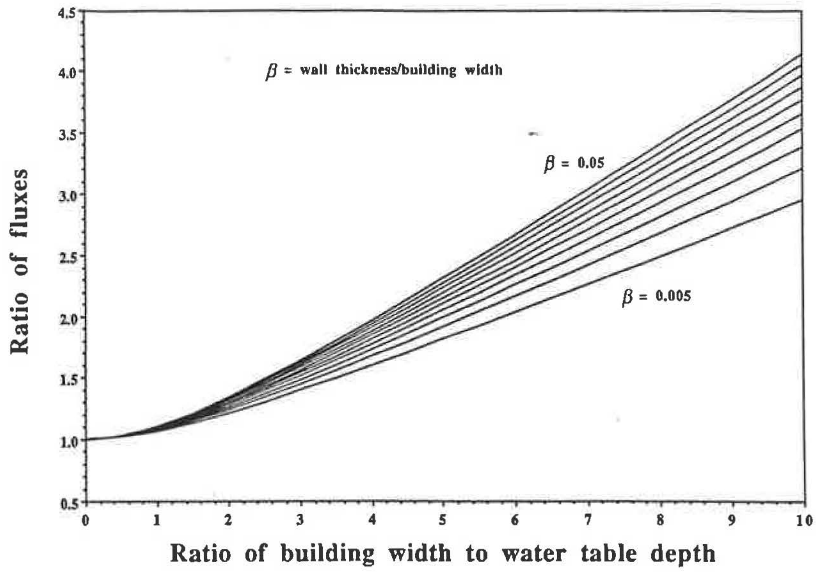

Fig. 4. Ratio of total flux from a floor with a water table at finite depth to the total flux in the absence of a water table, as a function of two parameters: the ratio of building width to water table depth, and β, the ratio of wall thickness to building width. Beta ranges from 0.005 to 0.05 in steps of 0.005. The water table temperature has been taken as zero, this is, equal to the outdoor temperature.

From (9), the total heat flux for a water table at inifinite depth is given by

$$\Phi_{\infty} = \frac{2k}{\pi}[\ln(1+1/\beta)+(1/\beta)\ln(1+\beta)]. \quad (11)$$

The flux ratio Φ/Φ_{∞} has been calculated for $0.1 \leqslant \gamma \leqslant 10.0$ and $0.05 \leqslant \beta \leqslant 0.005$ (for a building width of 10 m, these ranges correspond to a water table depth between 1 m and 100 m, and a wall thickness between 0.05 m and 0.5, which cover the dimensions of interest). The results are presented in Fig. 4, in which the water table temperature has been taken as zero, that is, equal to the outdoor temperature. This is a reasonable choice for the purposes of presenting the results graphically, although other values may well be equally realistic.

Figure 4 shows that for any water table depth greater than the building width ($\gamma \leqslant 1$), the increase in heat loss attributable to the water table is small, being less than 10%. Hagentoft [8] reached a similar conclusion via a somewhat different route, using a numerical analysis that took into account the ground water flow rate, the permeability of the soil, and insulation over the slab, with a step change in the surface temperature at the edge.

It is interesting to note that for constant γ the flux ratio increases with increasing β. This can be understood as follows: for a water table at infinite depth, an increase in β implies a decrease in building width at constant wall thickness, or an increase in wall thickness at constant building width; in either case the total flux will decrease to zero as β approaches infinity, since the second case can be rescaled to the first. For a water table at finite depth, an increase in β is equivalent to a decrease in building width *and* a decrease in water table depth by the same proportion, suggesting that the flux approaches a non-zero constant. This can be seen from (10), which shows that the flux approaches $k\gamma(1-T_w)$ as β approaches infinity; therefore the flux ratio increases. As β approaches zero, the flux ratio approaches one. This is because both (10) and (11) diverge at the same rate, since in this limit the surface temperature approaches a step change at the slab edge.

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