A Procedure for Calculating Thermal Response Factors of Multi-layer Walls—State Space Method

KUNZE OUYANG* FARIBORZ HAGHIGHAT†

Thermal response factors of building envelopes are fundamental information in the design of thermal systems. The conventional calculating method of these factors is first to calculate the eigenvalues of the system as a function of thermal physical properties and the thickness of materials. The roots finding process is time-consuming and occasionally may lead to miscalculation due to missing one or more roots. In this paper a new approach to calculate these factors based on the state space principle of modern control theory has been proposed. Using this approach through calculation of the state transition matrix of a system, the thermal response factors of a multi-layer wall can be obtained without finding the roots. This method is feasible and easy to use with a sufficient accuracy for engineering applications.

INTRODUCTION

THE THERMAL DYNAMIC characteristics of building envelopes, represented by s-transfer function coefficients or thermal response factors, are fundamental data for calculating air conditioning cooling loads and analysing annual energy consumption of buildings. The conventional method to calculate these data is first to calculate the eigenvalues of a system, i.e. the poles of s-transfer functions based on thermal physical properties and thickness of materials [1, 2]. The root finding process is computationally inefficient and occasionally may lead to miscalculation due to missing a root, particularly in the case where two adjacent roots are close together. Therefore, it is required to develop a new approach to calculate the thermal response factors.

Based on the state space principle of modern control theory a new method for calculating thermal response factors of building envelopes has been presented in this paper. Using this method, first the rational fractions of s-transfer function are obtained from simple series expansion. Then, the state space equation of the system is established and the state transition matrix is calculated. Finally, the response factors will be obtained by a series of matrices multiplications after the continuous system is discretized. The main advantages of this approach is that there is no need to find roots of the system. The calculation of an example has shown that this new method gives a sufficient accuracy for engineering applications.

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PROGRESSION EXPRESSION OF S-TRANSFER FUNCTIONS

Pipes’ [3] transmission matrix is the fundamental tool in the modelling of heat flow through building envelope. The matrix relates the temperatures and the heat flows on both sides of the envelope, which is in the form of:

$$\begin{bmatrix} \theta_o \\ \phi_o \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \theta_i \\ \phi_i \end{bmatrix}. \quad (1)$$

where $A$, $B$, $C$, and $D$ are transfer functions which characterize the dynamic thermal performance of a wall. $\theta$ and $\phi$ are the Laplace transforms of temperatures and heat fluxes, respectively. Subscripts o and i indicate outside and inside surfaces of the wall, respectively.

For a multi-layer wall, the transmission matrix of the whole wall is the product of the transmission matrices of each layer in the order in which they appear in the wall, i.e.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \times \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \times \cdots \times \begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix}. \quad (2)$$

The elements of the transmission matrices for each layer can be given either in hyperbolic functions or in complex exponential functions form:

$$A_i = D_i = \cosh (L_i \sqrt{s/a_i}),$$

$$B_i = R_i \sinh (L_i \sqrt{s/a_i}),$$

$$C_i = L_i \sqrt{s/a_i} \sinh (L_i \sqrt{s/a_i})/R_i,$$

where $L_i$ is the thickness of the $i$th layer; $a_i$ is the thermal
The system can be described by the state space equation:

\[
\begin{bmatrix}
\phi_0 \\
\phi_1
\end{bmatrix} = \begin{bmatrix} D/B & -1/B \\ 1/B & -A/B \end{bmatrix} \begin{bmatrix}
\theta_0 \\
\theta_1
\end{bmatrix},
\]

where \( D/B \) and \( A/B \) are called \( s \)-transfer functions. When \( \theta \) and \( \phi \) are expressed in \( s \)-transform (i.e. time series form), the \( s \)-transforms of the \( D/B \), \( 1/B \) and \( A/B \) are called \( s \)-transfer functions of the wall. They can be expressed as a ratio of two finite polynomials of \( s^{-1} \). For example:

\[
\frac{1}{B} = a_0 + a_1 s^{-1} + a_2 s^{-2} + \cdots + a_n s^{-n}
\]

The relationship between the \( s \)-transfer function coefficients and the thermal response factors is that thermal response factors of a wall can be obtained by applying the long division computation to equation (4).

All the elements of the matrix can be simply expanded in series of infinite powers of parameter \( s \), i.e.

\[
A_i = D_i = 1 + \left( \frac{L_i^2}{2l_i a_i} s^2 + \frac{L_i^4}{4l_i a_i^2} s^4 + \frac{L_i^6}{6l_i a_i^3} s^6 + \cdots \right)
\]

(5)

\[
B_i = R_i \left( \frac{L_i^2}{3l_i a_i} s^2 + \frac{L_i^4}{5l_i a_i^2} s^4 + \frac{L_i^6}{7l_i a_i^3} s^6 + \cdots \right)
\]

(6)

\[
C_i = \frac{1}{R_i} \left( \frac{L_i^2}{3l_i a_i} s^2 + \frac{L_i^4}{5l_i a_i^2} s^4 + \frac{L_i^6}{7l_i a_i^3} s^6 + \cdots \right)
\]

(7)

These series are consistently convergent and the rate of convergence is very high for all values of \( s \). The coefficients can be calculated when the dimensions and physical properties of various layers are known. The calculation of all the coefficients in the expansion series of the matrix elements can be carried out for the whole wall from matrices characteristic and series computations.

Consequently, the \( s \)-transfer functions of \( D/B \), \( 1/B \) and \( A/B \) for multi-layer slabs can be expressed in the form of rational fraction of polynomials. For instance:

\[
\frac{1}{B} = K \left( 1 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots + b_n s^n \right)
\]

(8)

and

\[
D/B = K \left( 1 + c_1 s + c_2 s^2 + c_3 s^3 + \cdots + c_n s^n \right)
\]

(9)

where \( K \) is the overall heat transfer coefficient. The required number of terms in polynomials, \( n \), depends on the accuracy needed.

### STATE EQUATION AND TRANSITION MATRIX

For a linear time-invariant continuous system with one input and one output, the dynamic characteristics of the system can be described by the state space equation:

\[
\dot{X}(t) = FX(t) + GU(t),
\]

(10)

and the output equation:

\[
Y(t) = h'X(t) + eU(t),
\]

(11)

where \( X(t) \) is the \( n \)-dimensional state variable of the system and \( \dot{X}(t) \) is the derivative of \( X(t) \) with respect to time \( t \). \( U(t) \) and \( Y(t) \) are input and output variables respectively. \( F \) is a \((n,n)\) constant matrix, \( G \) is a \((n,1)\) constant matrix, \( h' \) is a \( n \)-dimensional vector and \( e \) is constant [4-6] \( e \) denotes the matrix transpose operation.

It can be seen that the state equation of a system is the differential expression which describes the construction and properties of the system and that the output equation is only a transform of variables. It is also obvious that \( s \)-transfer functions of \( D/B \), \( 1/B \) and \( A/B \) have the same state equation but the coefficients of their output equations are different.

The elements of matrices and vectors of equations (10) and (11) can be obtained when \( s \)-transfer functions of a system are known as follows:

\[
F = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-1/b_{a_1} & -b_1/b_{a_1} & -b_2/b_{a_1} & \cdots & -b_{n-1}/b_{a_1}
\end{bmatrix}
\]

(12)

For \( 1/B \):

\[
h' = (1,0,0,\ldots,0) \quad e = 0.
\]

For \( D/B \):

\[
h' = (1-e,-c_1-b_1 e,\ldots,-c_{n-1}-b_{n-1} e) \quad e = c_{i}/b_{a_1}.
\]

(13)

The solution of the state equation (10) for a linear time-invariant continuous system can be expressed in the form of:

\[
X(t) = e^{Ft} X(0) + \int_0^t e^{(D-B) \tau} GU(\tau) d\tau,
\]

(15)

where \( X(0) \) is the initial state of the system and \( e^{Ft} \) is the state transition matrix of the system. The matrix exponential \( e^{Ft} \) can be calculated from the following expression:

\[
e^{Ft} = I + Ft + \frac{F^2 t^2}{2!} + \frac{F^3 t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{F^k t^k}{k!},
\]

(16)

where \( I \) is the identity matrix.

Particularly, when \( X(0) = 0 \), the output variable \( Y(t) \) of the system will be expressed in the form of:


\[ Y(t) = h^T \int_0^T e^{RT-\tau} GU(\tau) \, d\tau + e U(t) \]  

(17)

**DISCRETIZATION OF CONTINUOUS SYSTEMS**

To obtain the thermal response factors of a system, the discretized expression of variables in a continuous system can be used to simplify the determination [4, 6]. If first-order property holds, linear interpolation between the two adjacent instants can be used, and the state equation for the linear time-invariant discrete systems with single input and single output can be expressed as follows:

\[ X(k+1) = PX(k) + Q_1 U(k) + Q_2 U(k+1), \]  

(18)

where \( X(k) \) and \( X(k+1) \) describe the states of the system at \( k \)th and \((k+1)\)th instants, respectively. \( P \) is a constant \((n,n)\) matrix. \( Q_1 \) and \( Q_2 \) are the constant \((n,1)\) matrices. Equation (18) shows that the state of the system at the instant \( k+1 \) is not only related to the state of the system and input value at the previous time, but also to the input value at the present time. This is not realizable from control theory. In numerical calculation by computers, however, it is not difficult to solve equation (18) if the time sequence of input functions are known.

In comparing equations (15) and (18), we can see that the constant matrix \[ P = e^{RT} \]

where \( T \) is the time interval. The constant matrices \( Q_1 \) and \( Q_2 \) can be obtained by the following integration:

\[ Q_1, U(k) + Q_2, U(k+1) = \int_k^{k+1} e^{RT} \, d\tau. \]

Integrating the expression above, we obtain:

\[ Q_1 = \left[ \begin{array}{cccc} -b_1 & -b_2 & \cdots & -b_{n-1} & -b_n \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 1 & 0 \end{array} \right] \]

(19)

\[ Q_2 = \left[ \begin{array}{cccc} \frac{2F}{T} \left( e^{RT} - 1 \right) - F^{-1} \end{array} \right]. \]

(20)

Considering the output equation of the linear time-invariant discrete system, the output is:

\[ Y(k) = h^T X(k) + e U(k). \]

(21)

Given the time series of the input function, \( U(K) \), the time series of output function, \( Y(K) \), can be obtained from equations (18) and (21).

**CALCULATION OF THERMAL RESPONSE FACTORS**

When the input function \( U(k) \) is a triangular impulse, i.e., \( U(k) = (0, T, 0, 0, \ldots) \), the time series of output variable \( Y(k) \), which are the thermal response factors, can be determined. For instance, the heat transfer thermal response factors \( Y_{\text{HT}}(j) \) can be calculated as follows:

\[ Y_{\text{HT}}(0) = T(h^T Q_2 + e), \]

\[ Y_{\text{HT}}(1) = Th^T (Q_1 + PQ_2), \]

\[ = h^T F^{-2} (P^2 - 2P + 1) G, \]

\[ Y_{\text{HT}}(2) = Th^T PQ_2 \]

\[ = h^T F^{-2} (P^2 - 2P + 1) G. \]

When \( j > 1 \), the general expression of \( Y_{\text{HT}}(j) \) is:

\[ Y_{\text{HT}}(j) = h^T P^{-j} F^{-2} (P^2 - 2P + 1) G. \]

(22)

In the above equation it is required to calculate the inverse of matrix \( F^{-1} \). Since \( F \) is a companion matrix, its inverse \( F^{-1} \) is very simple and it can be directly given as follows:

\[ F^{-1} = \left[ \begin{array}{cccc} -b_1 & -b_2 & \cdots & -b_{n-1} & -b_n \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 1 & 0 \end{array} \right] \]

(23)

The transition matrix \( P = e^{RT} \) can be calculated from equation (16). When the elements \( f_{ij} \) of \( P \) and \( T \) are large it is possible to converge very slowly. In this case the square method can be used for accelerating the convergence. The calculating process is as follows:

Step 1: Calculate the norm of \( F \)

\[ ||F|| = \left( \sum_{i,j=1}^{n} f_{ij} \right)^{1/2} \]

(24)

and find a positive integer \( N \) to satisfy

\[ ||F||T < 2^N \]

(25)

Step 2: Calculate

\[ P_0 = I + \frac{FT}{2^N} + \frac{F^2T^2}{2!} + \frac{F^3T^3}{3!} + \cdots, \]

(26)

Step 3: Apply the following formula

\[ P_m = P_{m-1} P_{m-1} \quad (m = 1, 2, \ldots, N) \]

then \( P = P_N = e^{RT} \). It is convenient to computerise this method.

From above discussion we can see that once the state transition matrix \( P = e^{RT} \) is calculated, the response factors will be obtained by a series of matrix multiplications.

An example is given to illustrate the use of this approach for calculating the thermal response factors of a wall. The given wall consists of 5 layers of homogeneous materials including insulation between two layers of concrete with inside and outside air films. The dimensions
K. Ouyang and F. Haghighat

Table 1. The dimensions and physical properties of the wall

<table>
<thead>
<tr>
<th>No.</th>
<th>Thickness (m)</th>
<th>Thermal conditions (W m⁻¹°C⁻¹)</th>
<th>Density (kg m⁻³)</th>
<th>Heat capacity (J kg⁻¹°C⁻¹)</th>
<th>Resistance (m°C W⁻¹)</th>
<th>Material</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.089</td>
<td>1.73</td>
<td>2235</td>
<td>1106</td>
<td>0.05</td>
<td>Outside air</td>
</tr>
<tr>
<td>2</td>
<td>0.127</td>
<td>0.074</td>
<td>24</td>
<td>992</td>
<td>1.707</td>
<td>Concrete</td>
</tr>
<tr>
<td>3</td>
<td>0.089</td>
<td>1.73</td>
<td>2235</td>
<td>1106</td>
<td>0.0514</td>
<td>Insulation</td>
</tr>
<tr>
<td>4</td>
<td>0.127</td>
<td>0.074</td>
<td>24</td>
<td>992</td>
<td>1.707</td>
<td>Concrete</td>
</tr>
<tr>
<td>5</td>
<td>0.16</td>
<td></td>
<td></td>
<td></td>
<td>0.16</td>
<td>Inside air</td>
</tr>
</tbody>
</table>

Table 2. The expansion of the elements of transmission matrix

<table>
<thead>
<tr>
<th>Term</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00000000</td>
<td>1.00000000</td>
<td>0.00000000</td>
<td>1.00000000</td>
</tr>
<tr>
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<td>248.79390400</td>
<td>130.90036576</td>
</tr>
<tr>
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<td>258.2960749</td>
<td>56.5431391</td>
<td>1254.98859575</td>
<td>1322.04426585</td>
</tr>
<tr>
<td>3</td>
<td>625.0278451</td>
<td>56.37568714</td>
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</tr>
<tr>
<td>4</td>
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<td>9426.14335476</td>
<td>827.21983675</td>
</tr>
<tr>
<td>5</td>
<td>93.33127498</td>
<td>8.01690022</td>
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<td>254.33964197</td>
</tr>
<tr>
<td>6</td>
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<td>1.56014227</td>
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<tr>
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<td>0.06386258</td>
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<tr>
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<td>0.0011756</td>
<td>0.05054137</td>
<td>0.00473800</td>
</tr>
</tbody>
</table>

Table 3. The response factors for the given wall

<table>
<thead>
<tr>
<th>No.</th>
<th>State space method</th>
<th>Conventional method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00001771</td>
<td>0.0001549</td>
</tr>
<tr>
<td>1</td>
<td>0.00164078</td>
<td>0.00164541</td>
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<tr>
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<td>0.00852884</td>
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<tr>
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<td>0.01608904</td>
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<tr>
<td>4</td>
<td>0.02132861</td>
<td>0.02132482</td>
</tr>
<tr>
<td>5</td>
<td>0.02458189</td>
<td>0.02458376</td>
</tr>
<tr>
<td>6</td>
<td>0.02634117</td>
<td>0.02634535</td>
</tr>
<tr>
<td>7</td>
<td>0.02701426</td>
<td>0.02701681</td>
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<td>8</td>
<td>0.02690951</td>
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</tr>
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<td>0.02652429</td>
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</tr>
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<td>0.02398711</td>
</tr>
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<td>0.02258611</td>
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<tr>
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<td>0.01410310</td>
</tr>
<tr>
<td>19</td>
<td>0.01289871</td>
<td>0.01290017</td>
</tr>
</tbody>
</table>

Common ratio 0.91470457

and physical properties of the wall are given in Table 1. The overall heat transfer coefficient $K$ is 0.4951 W/M°C.

Using the data in Table 1, the elements in the matrix for each layer were calculated from equations (5)–(7). Then the overall transmission matrix of the wall was computed from matrices multiplication as given in Table 2.

Table 3 shows the thermal response factors of the wall for case $n = 8$ and $T = 1$ h obtained using the state space method. Comparing this result with that obtained by the conventional approach [6] which requires finding the roots of the system indicates this method provides sufficient accuracy (to five digits of decimal).

CONCLUSIONS

The state space method is developed to calculate thermal response factors of multilayer walls. One of the advantages of this method is that there is no need to find the roots of the system transfer function. Therefore, it may avoid miscalculation due to missing a root and the calculation process can be simplified.

The example of calculation has shown that a satisfactory result can be obtained by the state space method. It is necessary to do further research in order to get higher accuracy.
REFERENCES