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Moisture Diffusion due to Periodic Moisture and Temperature Boundary Conditions—an Approximate Steady Analytical Solution with Non-Constant Diffusion Coefficients

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An approximate steady analytical solution is given to the moisture diffusion equation with periodic moisture and temperature boundary conditions, where the moisture flux is determined by both temperature and moisture gradients, specifically $F = -D_m(\partial m/\partial x) - D_T(\partial T/\partial x)$ *and which has nonconstant diffusion coefficients. The key physical assumptions made are that the thermal conductivity is constant and that the Fourier number is* $\gg 1$ *so that the internal temperature gradient is linear but periodic in response to the periodic temperature boundary conditions. The key mathematical approximation made is to ignore second and higher harmonic terms in the time-dependent part of the solution. The solution agrees with a numerical model over a wide range of parameters to within 10%, and reduces in special cases to well known existing analytical solutions. The solution has applications in a wide number of moisture diffusion problems in building physics.*

NOMENCLATURE

- a coefficient of $\partial^2 m/\partial x^2$
- b coefficient of $\partial m/\partial x$
- c coefficient of $(\partial m/\partial x)^2$, also specific heat (J kg^{-1} °C^{-1})
- d term in differential equation
- D_m diffusion coefficient under moisture gradient (m^2 s^{-1})
- D_T diffusion coefficient under temperature gradient (kg m^{-1} s^{-1} °C^{-1})
- F moisture flux (kg m^{-2} s^{-1})
- Fo Fourier number (dimensionless)
- k thermal conductivity (W m^{-1} °C^{-1})
- l length (m)
- m moisture concentration (kg m^{-3})
- M moisture concentration (kg m^{-3})
- t time (s)
- T temperature (°C)
- x distance (m)
- θ phase of temperature boundary conditions (radians)
- μ coefficient of x
- ν coefficient of x^2
- τ dimensionless time
- ϕ phase of moisture concentration at $x = l$ (radians)
- ω angular frequency (radians s^{-1})

Subscripts

- 0 value at $x = 0$
- c coefficient of cos ωt
- l value at $x = l$
- n finite difference cell marker
- s coefficient of sin ωt
- ϕ, ψ potentials

Superscripts

- $-$ time average
- $=$ space and time average
- $'$ space derivative
- $''$ second derivative of space

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INTRODUCTION

THE DIFFUSION of moisture through materials is one of the important mechanisms mediating the transfer of moisture in building materials and structures. In some cases it is a dominant mechanism, e.g. in the drying of concrete, and in others is less important (e.g. many moisture issues in structures are dominated by air convective transfer). In all cases, however, a detailed understanding of this diffusive process is important if a deeper understanding of moisture transfer issues in building physics is sought.

A key diffusion problem is that of diffusion under periodic boundary conditions. Many problems of practical importance will take this form, with the major periods being one day and one year. Indeed, some problems that are solved for constant boundary conditions, e.g. dew-point profile calculations, are approximations to a true periodic boundary condition case. Furthermore, if the periodic solution is Fourier analysed then, providing the problem is sufficiently linear, the constant boundary condition case can be seen to be a special case. Some cases of importance in which moisture diffusion is driven by periodic moisture concentration and temperature boundary conditions are: walls and roofs without cavities, framing timber, concrete floors, insulation layers, moisture transfer in claddings and linings, and solar driven moisture transfer [1].

Most moisture transfer "in the field" takes place under conditions of temperature and moisture gradients, with boundary conditions that are usually approximately periodic with daily or yearly periods. Earlier work on moisture diffusion [2] tended to concentrate on a single-variable driving potential for moisture transfer, usually moisture concentration gradient or vapour pressure gradient. Under isothermal conditions these two can be related through the material's sorption curve. Under non-isothermal conditions, however, the diffusive driving potential is not well understood; much work recently (e.g. Kumuran *et al.* [3]) has been devoted to two-variable driving potentials to describe this situation, perhaps moisture concentration gradient and temperature gradient, or vapour pressure gradient and temperature gradient. The former is used in this paper although it is shown how other potentials can be derived from that chosen.

Whatever potentials are chosen, the corresponding diffusion coefficients are seldom constant. This can be seen in for example Tviet's data [4] and has been highlighted recently by Galbraith and McLean [5] in the case of vapour pressure as the potential. Since, in some cases, this variability can be quite large, any realistic modelling of moisture diffusion must take this variability into account.

Two-variable driving potentials with non-linear diffusion coefficients give rise to a complex diffusion equation which is not usually solved analytically, but rather numerically [6], possibly because an analytical solution is thought not to exist or to be too difficult to obtain. There are, however, a number of compelling reasons to seek an analytical solution. For example, an analytical result shows in one equation how the solution changes in response to the change in all the physical variables, whereas a numerical solution requires to be re-calculated for every change in the relevant variables. Consequently it is often possible to gain physical insight into the nature of the diffusive process by inspection of the analytical solution. As a corollary to this, sensitivity studies are, at least in principle, simply a matter of differentiation with respect to the relevant variable. Analytical formulae are fast and easy to evaluate and not affected by issues of convergence and accuracy that can plague numerical methods. Analytical solutions may allow special cases, including solutions already known, to be derived immediately. These solutions also allow for algebraic manipulation to enable the solution to be cast in a different form—e.g. to make the diffusion coefficients the subject of the formula.

To this end, this work begins by formulating a diffusion equation governing diffusion under moisture and temperature gradients with periodic boundary conditions, and solves for the steady solution under the conditions of constant thermal conductivity and large Fourier number, i.e. when the temperature gradient within the material, although still periodic, is always linear. Many problems in building physics belong in this class. Some special cases are examined and the general solution is checked against a numerical model. The work concludes by examining the issue of other driving potentials.

FORMULATION OF THE PROBLEM

Consider a material, length l, containing moisture diffusing under the driving potential

$$F = -D_m\frac{\partial m}{\partial x} - D_T\frac{\partial T}{\partial x}$$

and subject to periodic boundary conditions, viz.:

At $x = 0$

$$m(x=0) = M_0 = \bar{M}_0 + \delta M_0 \sin \omega t$$
$$T(x=0) = T_0 = \bar{T}_0 + \delta T_0 \sin(\omega t + \theta_0) \tag{1}$$

At $x = l$

$$m(x=l) = \bar{M}_l + \delta M_l \sin(\omega t + \phi)$$
$$T(x=l) = \bar{T}_l + \delta T_l \sin(\omega t + \theta_l) \tag{2}$$

The diffusion equation for this driving potential is:

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial x}\left(D_m\frac{\partial m}{\partial x} + D_T\frac{\partial T}{\partial x}\right)$$

i.e.

$$\frac{\partial m}{\partial t} = D_m\frac{\partial^2 m}{\partial x^2} + \left(\frac{\partial D_m}{\partial T} + \frac{\partial D_T}{\partial m}\right)\frac{\partial T}{\partial x}\frac{\partial m}{\partial x} + \frac{\partial D_m}{\partial m}\left(\frac{\partial m}{\partial x}\right)^2 + \frac{\partial D_T}{\partial T}\left(\frac{\partial T}{\partial x}\right)^2 + D_T\frac{\partial^2 T}{\partial x^2} \tag{3}$$

The corresponding heat diffusion equation is

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$$

assuming enthalpy advected by moisture movement is negligible.

Equation (3) in its full generality with moisture and temperature dependent diffusion coefficient D_T and D_m is not solvable analytically. However, we note first that in almost all situations of practical interest the time constant of response for temperature is very much shorter than for moisture concentration; furthermore we can assume for many cases that the Fourier number is $\gg 1$, i.e.

$$Fo = \frac{k}{\omega\rho c l^2} \gg 1$$

where $1/\omega$ is being taken as the characteristic time.

For example in wood of 10 cm thickness $Fo \sim 10^2$ for a 1 year period but $\sim 10^0$ for 1 day, while for fibreglass $Fo \sim 10^4$ and 10^2 for 1 year and 1 day periods respectively.

If further, the assumption is made that the thermal conductivity k is constant, then these two conditions have the important consequency that the temperature profile is linear but periodic, see for example Carlsaw and Jaeger [7].

Assuming that the Fourier number is large enough to maintain a periodic linear temperature gradient, the size of that gradient is given from the boundary conditions, equations (1) and (2) as follows:

$$\frac{\Delta T}{\Delta x} = \frac{1}{l}(\bar{T}_l - \bar{T}_0 + \delta T_l \sin(\omega t + \theta_l) - \delta T_0 \sin(\omega t + \theta_0))$$
$$= \frac{1}{l}(\overline{\Delta T} + \Delta T_s \sin \omega t + \Delta T_c \cos \omega t) \tag{4}$$

where

$$\overline{\Delta T} = \bar{T}_l - \bar{T}_0$$

$$\Delta T_s = \delta T_l \cos\theta_l - \delta T_0 \cos\theta_0$$

$$\Delta T_c = \delta T_l \sin\theta_l - \delta T_0 \sin\theta_0 \qquad (5)$$

Equation (3) under the boundary conditions (1) and (2) and the temperature field (4) constitutes the problem to be solved.

DIMENSIONLESS FORM

Equation (3) can be cast into a dimensionless form by making the following transformations

$$X = \frac{x}{l}, \quad M = \frac{m}{\bar{M}_0}, \quad \theta = \frac{T}{\bar{T}_0}, \quad \tau = \omega t.$$

This gives

$$\frac{\partial M}{\partial \tau} = A\frac{\partial^2 M}{\partial X^2} + B\frac{\partial M}{\partial X} + C\left(\frac{\partial M}{\partial X}\right)^2 + D \qquad (6)$$

where

$$A = \frac{D_m}{\omega l^2} \qquad (7)$$

$$B = \left(\frac{\partial D_m}{\partial T} + \frac{\partial D_T}{\partial m}\right)\frac{\Delta T}{\omega l^2} \qquad (8)$$

$$C = \frac{\bar{M}_0}{\omega l^2}\frac{\partial D_m}{\partial m} \qquad (9)$$

$$D = \frac{1}{\bar{M}_0 \omega}\frac{\partial D_T}{\partial T}\left(\frac{\Delta T}{l}\right)^2 \qquad (10)$$

The term arising from $D_T(\partial^2 T/\partial X^2)$ vanishes as $\partial T/\partial X$ is assumed constant, see equation (4).

The boundary conditions become

At $X = 0$

$$M(X=0) = 1 + \frac{\delta M_0}{\bar{M}_0}\sin\tau$$

$$\theta(X=0) = 1 + \frac{\delta T_0}{\bar{T}_0}\sin(\tau+\theta_0)$$

At $X = 1$

$$M(X=1) = \frac{\bar{M}_1}{\bar{M}_0} + \frac{\delta M_1}{\bar{M}_0}\sin(\tau+\phi)$$

$$\theta(X=1) = \frac{\bar{T}_1}{\bar{T}_0} + \frac{\delta T_1}{\bar{T}_0}\sin(\tau+\theta_1).$$

Aside from reference below to the dimensionless coefficients A–D, no further use will be made of this dimensionless form of the differential equation in this work.

SOLUTION METHOD

The broad outline only of the solution method will be given in this section. Details are given in the Appendix.

Equation (3) is first written in the form

$$\frac{\partial m}{\partial t} = a\frac{\partial^2 m}{\partial x^2} + b\frac{\partial m}{\partial x} + c\left(\frac{\partial m}{\partial x}\right)^2 + d$$

see equation (22) in the Appendix.

Since the solution is periodic, the approach followed and the approximations made are:

(1) The solution is written as a Fourier series as far as the first harmonic.
(2) Terms containing T in the above equation are expanded using equation (4).
(3) The terms $\partial D_m/\partial m$, $\partial D_m/\partial T$, $\partial D_T/\partial m$, $\partial D_T/\partial T$ are assumed to be constant.
(4) All terms in equation (3) are expanded as far as the first harmonic. In particular, it is shown that this requires only an average value of D_m need be considered.
(5) The non-time varying part of the expanded equation is solved approximately as a quadratic in distance x, by requiring the solution to satisfy the boundary conditions, and collocating at $x = \frac{1}{2}l$.
(6) The time varying first harmonic part of the expanded equation yields two second order simultaneous ordinary differential equations in the harmonic coefficients, which are solved by conventional methods.

The approximate solution is given in the Appendix as

$$m(x,t) = \bar{M}_0 + \mu x + \nu x^2 + m_s(x)\sin\omega t + m_c(x)\cos\omega t \qquad (11)$$

where μ is given by the appropriate root of the Appendix equation (42), ν is given by equation (41), and m_s and m_c are given by equations (48) and (49), respectively.

To illustrate the nature of these solutions, consider the very common situation of a building product or element with an oscillating temperature gradient across it, and an opposing oscillating moisture gradient, e.g. a yearly cyclic temperature gradient across a wall with cyclic relative humidity boundary conditions that are maximum when the temperature is minimum and vice-versa. Boundary conditions of this sort are specified in four cases in Table 1 which also contains a range of material parameters that could describe wood or mineral wool. The data in this table are meant for illustrative purposes only but are based on data in Tviet [4], or estimated. Note that D_T does not feature in the table because it plays no part in the solution, see comments in the Appendix. Figures 1–4 illustrate the resulting moisture profiles as calculated by formula (11). Shown in these figures are the moisture profiles at increments of 1/8th of a period.

It can be seen that for Wood I, Fig. 1, the moisture content in the middle of the material has an amplitude that is roughly equal to the mean of the driving amplitudes. In other words, moisture deep within this material feels almost the entire driving forces. This is not so for the other cases which show various degrees of damping in the amplitude of the moisture response in the centre of the material. This behaviour follows because Wood I has the highest value of the dimensionless coefficient A, equation (7), which governs the degree of damping with depth of the amplitude of the harmonic component of the moisture concentration.

All cases have a moisture concentration profile which

Table 1. Coefficients and boundary conditions for selected cases (see text)

Quantity	Units	Material type Wood I	Wood II	Mineral wool I	Mineral wool II
D_m	m^2/s	5.8×10^{-10}	1.7×10^{-10}	5.1×10^{-8}	5.1×10^{-8}
$(\partial D_m/\partial T)+(\partial D_T/\partial m)$	m^2/s °C	2.5×10^{-10}	1.0×10^{-10}	7.3×10^{-9}	4.6×10^{-9}
$\partial D_m/\partial m$	m^5/kg s	5.0×10^{-12}	1.0×10^{-11}	1.0×10^{-8}	1.0×10^{-8}
$\partial D_T/\partial T$	kg/m s °C^2	1.5×10^{-10}	1.0×10^{-10}	4.5×10^{-10}	4.5×10^{-10}
$\bar{M}_0$	kg/m^3	48	48	5.0	5.0
$\bar{M}_l$	kg/m^3	72	72	7.5	7.5
δM_0	kg/m^3	12	12	1.25	1.25
δM_l	kg/m^3	16	32	1.5	1.5
ϕ	radians	0	0	0	0
$\overline{\Delta T}_0$	°C	−8	−8	−8	−8
ΔT_s	°C	−2	−2	−2	−2
ΔT_c	°C	0	0	0	0
l	m	0.1	0.1	0.1	0.1
Period		1 year	1 year	1 day	1 day
Figure		1	2	3	4

tends to be concave upwards. Two conflicting effects are in action here: at the high moisture concentration end of the material D_T is larger due to the positive value of $\partial D_T/\partial m$ so the temperature gradient drives moisture into this region tending to cause the profile to be concave; however, at the other end of the material the temperature is higher in these examples so D_T is larger there due to the positive value of $\partial D_T/\partial T$ which tends to cause the profile to be convex. The ratio B/D determines which effect dominates, where B and D are dimensionless coefficients given by equations (8) and (10) respectively. In these examples this ratio is much larger than 1, implying that the profile will tend to be concave.

SPECIAL CASES

Case 1

$$(\bar{b}+2c\bar{m}')^2 \ll 4\bar{a}\omega$$

where $\bar{a}$, $\bar{b}$, $\bar{m}'$, are mean values of a, b and m' defined by equations (36), (30), and (34) respectively in the Appendix.

This case occurs at higher frequencies, e.g. 1 day, or when the effects of $\partial D_m/\partial T$, $\partial D_T/\partial m$ and $\partial D_m/\partial m$ are small. In this case

$$\alpha_1 = -\alpha_2 = \beta = \sqrt{\frac{\omega}{2\bar{a}}}$$

The solution, equations (48) and (49) in the Appendix, reduces to

$$\begin{aligned} m_s = \frac{1}{\Delta}\{ & W\cosh\beta x\cos\beta x(\cos 2\beta l-\cosh 2\beta l) \\ & +\sinh\beta x\cos\beta x(W\sinh 2\beta l+X\sin 2\beta l \\ & -2Y\sinh\beta l\cos\beta l-2Z\cosh\beta l\sin\beta l) \\ & +\cosh\beta x\sin\beta x(W\sin 2\beta l-X\sinh 2\beta l \\ & -2Y\cosh\beta l\sin\beta l+2Z\sinh\beta l\cos\beta l) \\ & +X\sinh\beta x\sin\beta x(\cosh 2\beta l-\cos 2\beta l)\} \\ & +\frac{b_c\bar{m}'+d_c}{\omega} \end{aligned}$$

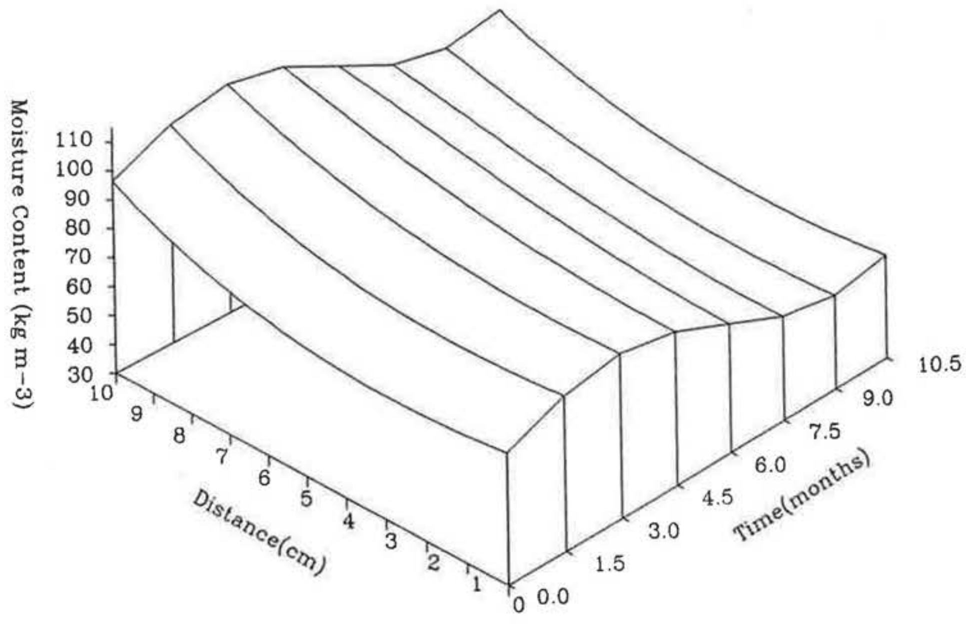

Fig. 1. Moisture content profiles at increments of 1/8 of a period for wood I.

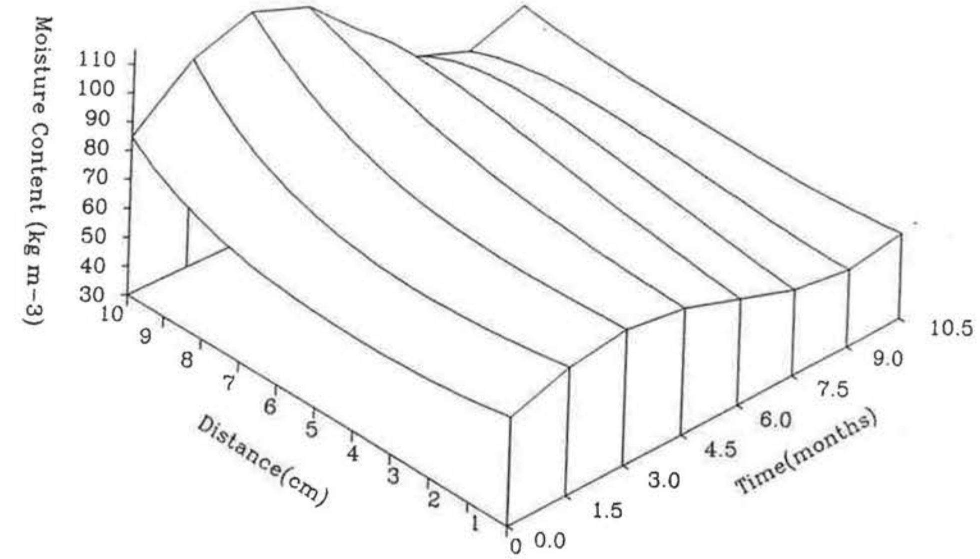

Fig. 2. Moisture content profiles at increments of 1/8 of a period for wood II.

and

$$m_c = \frac{1}{\Delta}\{X \cosh \beta x \cos \beta x (\cos 2\beta l - \cosh 2\beta l)$$
$$+ \sinh \beta x \cos \beta x(-W \sin 2\beta l + X \sinh 2\beta l$$
$$+ 2Y \cosh \beta l \sin \beta l - 2Z \sinh \beta l \cos \beta l)$$
$$+ \cosh \beta x \sin \beta x(W \sinh 2\beta l + X \sinh 2\beta l$$
$$- 2Y \sinh \beta l \cos \beta l - 2Z \cosh \beta l \sin \beta l)$$
$$+ W \sinh \beta x \sin \beta x(\cos 2\beta l - \cosh 2\beta l)\}$$
$$- \frac{b_s \bar{m}' + d_s}{\omega}$$

where Δ is now given by

$$\Delta = 2(\cos^2 \beta l - \cosh^2 \beta l)$$

and W, X, Y and Z are given by equation (47) in the Appendix.

Case 2

Boundary conditions identical on both sides and, no non-linear effects, i.e. no imposed temperature and moisture gradients, $\partial D_m/\partial m = 0$, and identical oscillating moisture boundary contents at $x = 0$ and $x = l$, i.e.

$$\bar{b} = c = d = \phi = 0, \quad \bar{M}_0 = \bar{M}_l, \quad \text{and} \quad \delta M_0 = \delta M_l$$

This is equivalent to the periodic boundary condition case analysed in Carslaw and Jaeger [7].

In this case the non time-dependent part of the solution vanishes and the time-dependent solution becomes

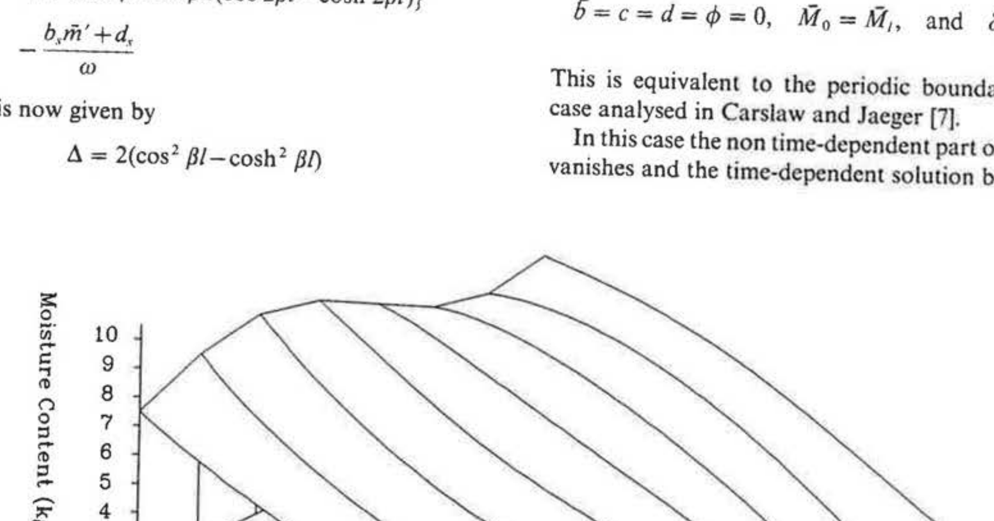

Fig. 3. Moisture content profiles at increments of 1/8 of a period for mineral wool I.

$$m_s = \delta M_0\left[\frac{(\cos\beta l+\cosh\beta l)\cosh\beta x\cos\beta x-\sinh\beta l\sinh\beta x\cos\beta x+\sin\beta l\cosh\beta x\sin\beta x}{\cosh\beta l+\cos\beta l}\right]$$

and

$$m_c = \delta M_0\left[\frac{(\cos\beta l+\cosh\beta l)\sinh\beta x\sin\beta x-\sin\beta l\sinh\beta x\cos\beta x-\sinh\beta l\cosh\beta x\sin\beta x}{\cosh\beta l+\cos\beta l}\right]$$

which can be shown to be identical to Carslaw and Jaeger's steady solution.

COMPARISON TO A NUMERICAL MODEL

The solution, equations (48) and (49) in the Appendix, is complex enough to justify checking against a numerical model. This comparison also allows an assessment of the inaccuracies incurred as a result of the approximations made.

A finite-difference numerical scheme was used with the following difference scheme

$$\frac{m'_n-m_n}{\Delta t} = a\frac{(m_{n+1}-2m_n+m_{n-1})}{(\Delta x)^2}+b\frac{(m_{n+1}-m_{n-1})}{2\Delta x}+c\frac{(m_{n+1}-m_n)(m_n-m_{n-1})}{(\Delta x)^2}+d$$

i.e.

$$\frac{m'_n-m_n}{\Delta t} = m_{n+1}\left(\frac{a}{(\Delta x)^2}+\frac{b}{2\Delta x}+\frac{c(m_n-m_{n-1})}{2(\Delta x)^2}\right)+m_n\left(-\frac{2a}{(\Delta x)^2}+\frac{c(m_{n+1}-2m_n+m_{n-1})}{2(\Delta x)^2}\right)+m_{n-1}\left(\frac{a}{(\Delta x)^2}-\frac{b}{2\Delta x}+\frac{c(m_n-m_{n+1})}{2(\Delta x)^2}\right)+d \quad (12)$$

This scheme was suitably modified at the boundaries. The numerical scheme was solved explicitly, with boundary conditions, temperature fields, and a (being moisture and temperature dependent) being updated at each time-step.

The results of the numerical calculation, taken after sufficient cycles had passed to allow any transients to die away, were compared with the approximate analytical result, formula (11) using an error term defined as

$$\text{error}(x) = (\text{numerical result}(x)-\text{analytical result}(x))/0.5(\bar{M}_0+\bar{M}_l)$$

Using this criterion, the largest error found in the cases examined was -9.2%, which occurred under conditions in which the size of D_m would vary by a factor of over 100. This case is illustrated in Fig. 5. Figure 6 illustrates a less extreme case where the error is -4.0%. Parameters describing the conditions used to achieve the moisture profiles shown in Figures 5 and 6 are contained in Table 2. Both cases were taken at the beginning of a period.

OTHER DRIVING POTENTIALS

Although moisture and temperature gradients are used in this work as the driving potentials, it is straightforward to extend the results to other driving potentials as the following analysis shows.

Any driving potential expressed through a pair of variables can be expressed in terms of any other pair of variables provided an equation of state exists connecting the variables involved together, e.g. if the moisture flux, F is expressed as

$$F = -D_m\frac{\partial m}{\partial x}-D_T\frac{\partial T}{\partial x} \quad (13)$$

or as

$$F = -D_\phi\frac{\partial \phi}{\partial x}-D_\psi\frac{\partial \psi}{\partial x} \quad (14)$$

then equation (14) can be rewritten as

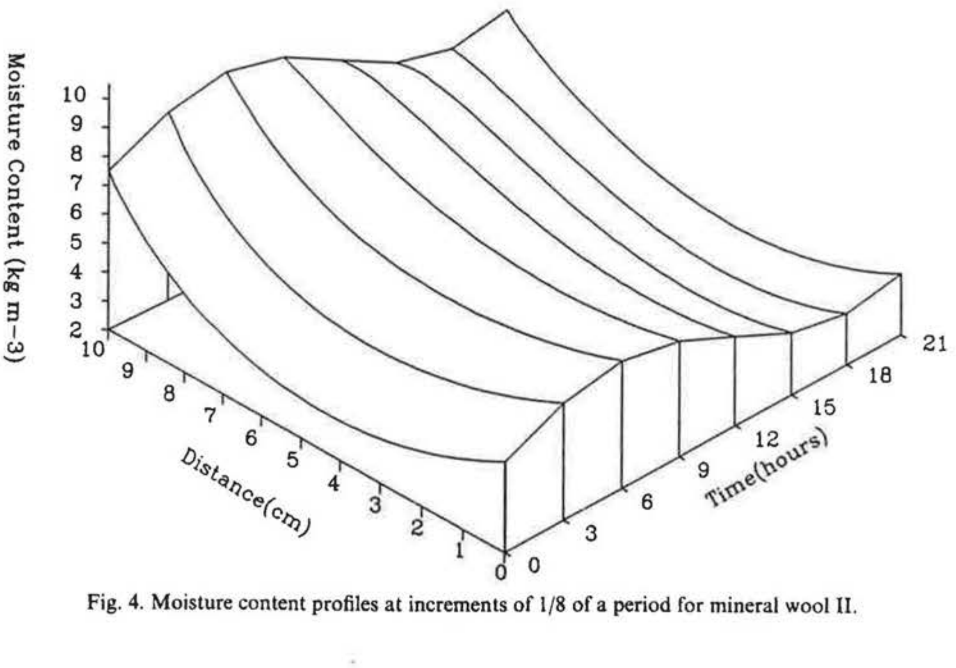

Fig. 4. Moisture content profiles at increments of 1/8 of a period for mineral wool II.

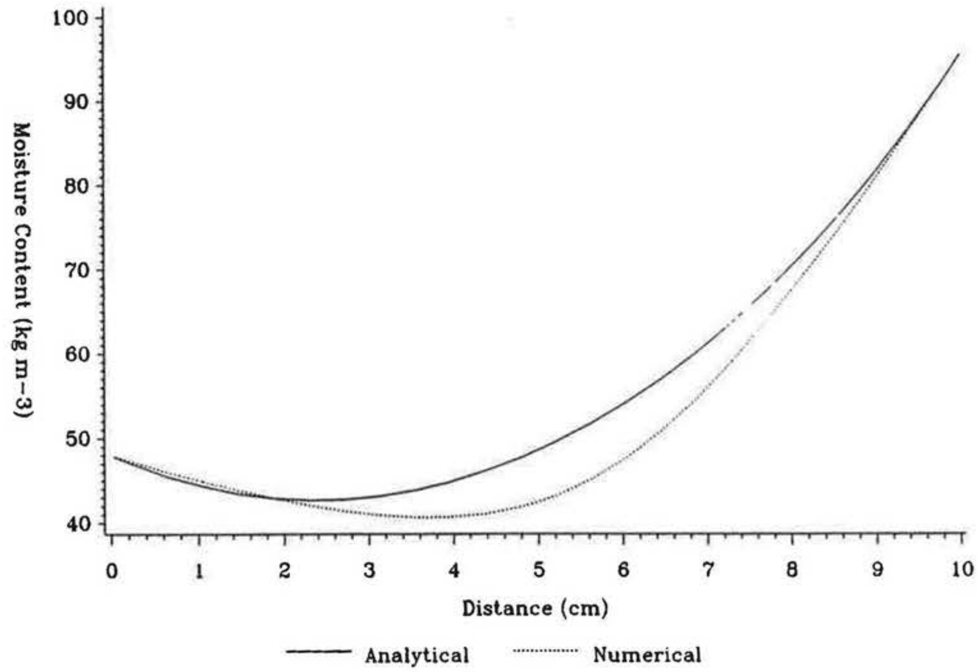

Fig. 5. Comparison between the analytical and numerical results. Maximum error −9.2%.

$$F = -D_\phi\left(\frac{\partial\phi}{\partial m}\frac{\partial m}{\partial x} + \frac{\partial\phi}{\partial T}\frac{\partial T}{\partial x}\right) - D_\psi\left(\frac{\partial\psi}{\partial m}\frac{\partial m}{\partial x} + \frac{\partial\psi}{\partial T}\frac{\partial T}{\partial x}\right)$$
$$= -\left(D_\phi\frac{\partial\phi}{\partial m} + D_\psi\frac{\partial\psi}{\partial m}\right)\frac{\partial m}{\partial x} - \left(D_\phi\frac{\partial\phi}{\partial T} + D_\psi\frac{\partial\psi}{\partial T}\right)\frac{\partial T}{\partial x} \tag{15}$$

so that by comparison between equations (13) and (15)

$$D_m = D_\phi\frac{\partial\phi}{\partial m} + D_\psi\frac{\partial\psi}{\partial m}, \quad D_T = D_\phi\frac{\partial\phi}{\partial T} + D_\psi\frac{\partial\psi}{\partial T}. \tag{16}$$

If, for example we wished to use the driving potentials of vapour pressure and temperature gradients, i.e.

$$F = -E_p\frac{\partial p}{\partial x} - E_T\frac{\partial T}{\partial x} \tag{17}$$

then from equation (16)

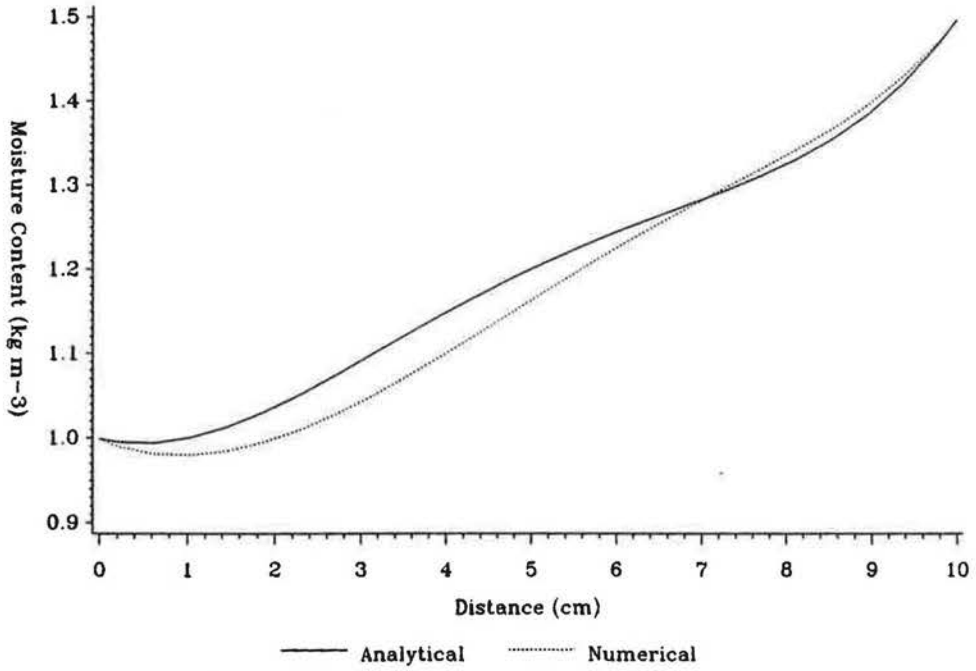

Fig. 6. Comparison between the analytical and numerical results. Maximum error −4.0%.

Table 2. Coefficients, boundary conditions and errors for numerical versus analytical comparisons (see text)

Quantities	Units	Values Example I	Example II
D_m	m^2/s	2.0×10^{-10}	2.9×10^{-8}
$(\partial D_m/\partial T)+(\partial D_T/\partial m)$	$m^2/s\ ^\circ C$	7.5×10^{-11}	2.7×10^{-9}
$\partial D_m/\partial m$	$m^5/kg\ s$	8.3×10^{-12}	1.5×10^{-8}
$\partial D_T/\partial T$	$kg/m\ s\ ^\circ C^2$	4.5×10^{-12}	1.1×10^{-9}
$\bar{M}_0$	kg/m^3	48	1.0
$\bar{M}_l$	kg/m^3	96	1.5
δM_0	kg/m^3	14.4	0.25
δM_l	kg/m^3	14.4	0.3
ϕ	radians	0	0
$\overline{\Delta T}_0$	°C	−8	−8
ΔT_s	°C	−0.8	−0.8
ΔT_c	°C	0	0
l	m	0.1	0.1
Period		1 year	1 day
Error	%	−9.2	−4.0
Figure		5	6

$$D_m = E_p \frac{\partial p}{\partial m}$$

and

$$D_T = E_T + E_p \frac{\partial p}{\partial T}$$

$\partial p/\partial T$ can be found from the equation of state for saturated water vapour pressure, and $\partial p/\partial m$ can be found from the sorption curve of the material in question.

The principle is the same above the hygroscopic region. Here the potentials often used are related to pore pressure, e.g. suction or chemical potential or other. Although these sorts of potentials are extremely sensitive to the exact value of the relative humidity within the material, their values change in a monotonic and non-extreme way as a function of moisture content, e.g. Siau [8] shows moisture content changing by a factor of 4 (30% to 120% by weight) as water potential ψ changes by a factor of 10 (−0.1 atmospheres to −0.01 atmospheres).

CONCLUSION

An approximate steady analytical solution has been given to the moisture diffusion equation with periodic moisture and temperature boundary conditions, where the moisture flux is determined by both temperature and moisture gradients and the diffusion coefficients are non-constant. This problem can be argued as a principal diffusion problem in building physics. Although the key mathematical approximation made was to ignore second and higher harmonics in the solution, checking of the results against a numerical model showed that the difference between the numerical and analytical model were less than 10% in all cases considered, even for quite extreme cases where the diffusion coefficient varies by over a factor of 100 due to its temperature and moisture variability.

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APPENDIX
SOLUTION METHOD IN DETAIL HARMONIC EXPANSION OF THE DIFFERENTIAL EQUATION

The equation to be solved is

$$\frac{\partial m}{\partial t} = D_m \frac{\partial^2 m}{\partial x^2} + \left(\frac{\partial D_m}{\partial T} + \frac{\partial D_T}{\partial m}\right)\frac{\partial T}{\partial x}\frac{\partial m}{\partial x} + \frac{\partial D_m}{\partial m}\left(\frac{\partial m}{\partial x}\right)^2 + \frac{\partial D_T}{\partial T}\left(\frac{\partial T}{\partial x}\right)^2 + D_T \frac{\partial^2 T}{\partial x^2} \quad (18)$$

subject to the periodic boundary conditions

At $x = 0$

$$m(x=0) = M_0 = \bar{M}_0 + \delta M_0 \sin \omega t$$
$$T(x=0) = T_0 = \bar{T}_0 + \delta T_0 \sin(\omega t + \theta_0) \quad (19)$$

At $x = l$

$$m(x=l) = \bar{M}_l + \delta M_l \sin(\omega t + \phi)$$
$$T(x=l) = \bar{T}_l + \delta T_l \sin(\omega t + \theta_l) \quad (20)$$

with a background linear periodic temperature field given by

$$\frac{\Delta T}{\Delta x} = \frac{1}{l}(\overline{\Delta T} + \Delta T_s \sin \omega t + \Delta T_c \cos \omega t) \quad (21)$$

Equation (18) will be written as

$$\frac{\partial m}{\partial t} = a\frac{\partial^2 m}{\partial x^2} + b\frac{\partial m}{\partial x} + c\left(\frac{\partial m}{\partial x}\right)^2 + d \quad (22)$$

where

$$a = D_m \quad (23)$$

$$b = \left(\frac{\partial D_m}{\partial T} + \frac{\partial D_T}{\partial m}\right)\frac{\partial T}{\partial x} = \left(\frac{\partial D_m}{\partial T} + \frac{\partial D_T}{\partial m}\right)\frac{\Delta T}{l} \quad (24)$$

$$c = \frac{\partial D_m}{\partial m} \quad (25)$$

$$d = \frac{\partial D_T}{\partial T}\left(\frac{\partial T}{\partial x}\right)^2 = \frac{\partial D_T}{\partial T}\left(\frac{\Delta T}{l}\right)^2. \quad (26)$$

The term $D_T(\partial^2 T/\partial x^2)$ vanishes as $\partial T/\partial x$ is assumed constant, see equation (21).

Since D_m is a function of T and m, its value will vary with position x and time t. Since the steady solution is periodic then the variation of D_m at each position is also periodic and consequently can be Fourier analysed. Expanding D_m to the first harmonic at each x we have

$$a = D_m(x,t) \simeq \bar{D}(x) + D_s(x)\sin\omega t + D_c(x)\cos\omega t$$

where $\bar{D}(x)$ is the mean of $D_m(x,t)$ over one time period. But

$$D_m(x) = D_m(m,T) \simeq D_m(T_0, M_0) + \frac{\partial D_m}{\partial T}(T-T_0) + \frac{\partial D_m}{\partial m}(m-M_0) \quad (27)$$

with exact equality if $\partial D_m/\partial T$ and $\partial D_m/\partial m$ are constant, i.e. $D_m(x)$ is linear in T and m.

Under these conditions, since for a linear function, the mean value of the function is equal to the value at the means of its arguments, then

$$\bar{D}(x) = D_m(\bar{m}(x), \bar{T}(x)) = D_m(\bar{m}(x), \bar{T}_0 + x\overline{\Delta T}/l) = \bar{a}$$

where

$$\bar{a} = D_m(\bar{m}(x), \bar{T}_0 + x\overline{\Delta T}/l)$$

and $\bar{m}(x)$ and $\bar{T}(x)$ are the mean values over one period of m and T respectively.

Hence

$$a = D_m(x,t) \simeq \bar{a}(x) + D_s(x)\sin\omega t + D_c(x)\cos\omega t. \quad (28)$$

For b we have

$$\begin{aligned} b &= \left(\frac{\partial D_m}{\partial T} + \frac{\partial D_T}{\partial m}\right)\frac{\partial T}{\partial x} \\ &= \frac{1}{l}\left(\frac{\partial D_m}{\partial T} + \frac{\partial D_T}{\partial m}\right)(\overline{\Delta T} + \Delta T_s \sin\omega t + \Delta T_c \cos\omega t) \\ &= \bar{b} + b_s\sin\omega t + b_c\cos\omega t \end{aligned} \quad (29)$$

where

$$\bar{b} = \frac{\overline{\Delta T}}{l}\left(\frac{\partial D_m}{\partial T} + \frac{\partial D_T}{\partial m}\right) \quad (30)$$

$$b_s = \frac{\Delta T_s}{l}\left(\frac{\partial D_m}{\partial T} + \frac{\partial D_T}{\partial m}\right) \quad (31)$$

$$b_c = \frac{\Delta T_c}{l}\left(\frac{\partial D_m}{\partial T} + \frac{\partial D_T}{\partial m}\right) \quad (32)$$

The boundary conditions are periodic and hence the steady solution is also periodic and can therefore be Fourier analysed. The key mathematical approximation is now made of expanding this periodic steady solution as far as the first harmonic only, i.e. m is taken as

$$m = \bar{m} + m_s\sin\omega t + m_c\cos\omega t \quad (33)$$

$\partial m/\partial x$ and $\partial^2 m/\partial x^2$ are therefore given by

$$\frac{\partial m}{\partial x} = \bar{m}' + m_s'\sin\omega t + m_c'\cos\omega t \quad (34)$$

$$\frac{\partial^2 m}{\partial x^2} = \bar{m}'' + m_s''\sin\omega t + m_c''\cos\omega t \quad (35)$$

Expanding each term in the differential equation (22) we have from equations (28) and (35)

$$\begin{aligned} a\frac{\partial^2 m}{\partial x^2} &= (\bar{a} + D_s(x)\sin\omega t + D_c(x)\cos\omega t) \\ &\quad\times(\bar{m}'' + m_s''\sin\omega t + m_c''\cos\omega t) \\ &= \bar{a}\bar{m}'' + \tfrac{1}{2}D_s m_s'' + \tfrac{1}{2}D_c m_c'' + (\bar{a}m_s'' + \bar{m}''D_s)\sin\omega t \\ &\quad + (\bar{a}m_c'' + \bar{m}''D_c)\cos\omega t + \textit{2nd harmonic terms} \\ &\simeq \bar{\bar{a}}(\bar{m}'' + m_s''\sin\omega t + m_c''\cos\omega t) \end{aligned}$$

where

$$\bar{\bar{a}} = \bar{D}(l/2) = D_m(\bar{m}(l/2), \bar{T}_0 + \overline{\Delta T}/2) \quad (36)$$

i.e. ignoring second derivatives of m_s and m_c in the non time-dependent part of the expansion, and the second derivative of $\bar{m}$ in the time-dependent part, and approximating $\bar{a}$ with its value at $x = l/2$. These approximations are of the same order or higher than approximations made below: in equation (37) the first derivatives of m_s and m_c are ignored in the non time-dependent part of the expansion; and the first derivative of $\bar{m}$ is taken as constant in the time-dependent part of the expansion in equation (43).

Expanding $b(\partial m/\partial x)$ in equation (22) using equation (29) and (34) gives

$$\begin{aligned} b\frac{\partial m}{\partial x} &= (\bar{b} + b_s\sin\omega t + b_c\cos\omega t) \\ &\quad\times(\bar{m}' + m_s'\sin\omega t + m_c'\cos\omega t) \\ &= \bar{b}\bar{m}' + \tfrac{1}{2}(b_s m_s' + b_c m_c') + (\bar{b}m_s' + b_s\bar{m}')\sin\omega t \\ &\quad + (\bar{b}m_c' + b_c\bar{m}')\cos\omega t + \textit{2nd harmonic terms} \end{aligned}$$

Also from equation (34)

$$\begin{aligned} c\left(\frac{\partial m}{\partial x}\right)^2 &= c(\bar{m} + m_s'\sin\omega t + m_c'\cos\omega t)^2 \\ &= c(\bar{m}'^2 + \tfrac{1}{2}(m_s'^2 + m_c'^2) + 2\bar{m}'m_s'\sin\omega t \\ &\quad + 2\bar{m}'m_c'\cos\omega t + \textit{2nd harmonic terms}) \end{aligned}$$

and from equations (26) and (21)

$$\begin{aligned} d &= \frac{\partial D_T}{\partial T}\left(\frac{\partial T}{\partial x}\right)^2 \\ &= \frac{1}{l^2}\frac{\partial D_T}{\partial T}(\overline{\Delta T}^2 + \tfrac{1}{2}\Delta T_s^2 + \tfrac{1}{2}\Delta T_c^2 \\ &\quad + 2\overline{\Delta T}\Delta T_s\sin\omega t + 2\overline{\Delta T}\Delta T_c\cos\omega t + \textit{2nd harmonic terms}) \\ &\simeq d + d_s\sin\omega t + d_c\cos\omega t \end{aligned}$$

where

$$\bar{d} = \frac{1}{l^2}\frac{\partial D_T}{\partial T}(\overline{\Delta T}^2 + \tfrac{1}{2}\Delta T_s^2 + \tfrac{1}{2}\Delta T_c^2)$$

$$d_s = 2\frac{\overline{\Delta T}\Delta T_s}{l^2}\frac{\partial D_T}{\partial T}$$

$$d_c = 2\frac{\overline{\Delta T}\Delta T_c}{l^2}\frac{\partial D_T}{\partial T}$$

while from equation (33)

$$\frac{\partial m}{\partial t} = \omega m_s\cos\omega t - \omega m_c\sin\omega t.$$

Substituting for the expressions for a, b, c, d, $\partial m/\partial t$ in equation (22) and comparing non time-dependent parts of the expansion gives

$$\bar{\bar{a}}\bar{m}'' + \bar{b}\bar{m}' + \tfrac{1}{2}(b_s m_s' + b_c m_c') + c(\bar{m}'^2 + \tfrac{1}{2}(m_s'^2 + m_c'^2)) + \bar{d} = 0$$

i.e.

$$\bar{\bar{a}}\bar{m}'' + \bar{b}\bar{m}' + c\bar{m}'^2 + \bar{d} \simeq 0 \quad (37)$$

ignoring m_s' and m_c'.

Comparing sin and cos terms in the expansion gives

$$\bar{\bar{a}}m_s'' + (\bar{b} + 2c\bar{m}')m_s' + b_s\bar{m}' = -\omega m_c - d_s \quad (38)$$

$$\bar{\bar{a}}m_c'' + (\bar{b} + 2c\bar{m}')m_c' + b_c\bar{m}' = \omega m_s - d_c \quad (39)$$

SOLUTION TO THE NON TIME-DEPENDENT PART OF THE EQUATION

Equation (37) is solved approximately as a quadratic in distance x, by requiring the solution to satisfy the boundary conditions, and collocating at $x = \frac{1}{2}l$.

In order that the quadratic

$$\bar{m} = \bar{M}_0 + \mu x + \nu x^2 \tag{40}$$

satisfy the boundary conditions

at $x = 0$

$$\bar{m}(x=0) = \bar{M}_0$$

and at $x = l$

$$\bar{m}(x=l) = \bar{M}_l$$

we will have

$$\nu = \frac{\frac{\Delta M}{l} - \mu}{l} \tag{41}$$

where

$$\Delta M = \bar{M}_l - \bar{M}_0$$

Collocating the quadratic at $x = l/2$ requires the solution to be exact at that point.

At $x = l/2$ we have from equation (40)

$$\bar{m} = \bar{M}_0 + \frac{\mu l}{4} + \frac{\Delta M}{4}$$

$$\bar{m}' = \frac{\Delta M}{l}$$

$$\bar{m}'' = 2\left(\frac{\frac{\Delta M}{l} - \mu}{l}\right)$$

$$\begin{aligned}\bar{\bar{a}} &= D_m(m = \bar{M}_0 + \Delta M/4 + \mu l/4, T = \bar{T}_0 + \overline{\Delta T}/2) \\ &= D_m(m = \bar{M}_0 + \Delta M/4, T = \bar{T}_0 + \overline{\Delta T}/2) + \frac{\partial D_m}{\partial m}\frac{\mu l}{4}\end{aligned}$$

from equation (27). Hence

$$\bar{\bar{a}} = \bar{a} + c\frac{\mu l}{4}$$

where

$$\bar{a} = D_m(m = \bar{M}_0 + \Delta M/4, T = \bar{T}_0 + \overline{\Delta T}/2)$$

Collocation at $x = l/2$ using equation (37) gives

$$\frac{c}{2}\mu^2 + \left(\frac{2\bar{a}}{l} - \frac{c}{2}\frac{\Delta M}{l}\right)\mu - \frac{\Delta M}{l}\left(\frac{2\bar{a}}{l} + \bar{b} + \frac{c\Delta M}{l}\right) - \bar{d} = 0. \tag{42}$$

If $c = 0$

$$\mu = \frac{\Delta M}{l}\left(1 + \frac{l\bar{b}}{2\bar{a}}\right) + \frac{l\bar{d}}{2\bar{a}}.$$

If $c \neq 0$ then the quadratic, equation (42) is solved using the physical root found by taking the plus sign in the standard quadratic solution formula.

SOLUTION TO THE TIME-DEPENDENT PART OF THE EQUATION

Elimination between (38) and (39) gives

$$\bar{\bar{a}}^2 m_s'''' + 2\bar{\bar{a}}(\bar{b} + 2c\bar{m}')m_s''' + (\bar{b} + 2c\bar{m}')^2 m_s'' + \omega^2 m_s - \omega(b_c\bar{m}' + d_c) = 0 \tag{43}$$

where $\bar{m}'$ has been taken as approximately constant, see equation (45), so that $\bar{m}''$ and $\bar{m}'''$ can be taken as zero.

The solution to equation (43) is

$$m_s = A\,e^{(\alpha_1 + j\beta)x} + B\,e^{(\alpha_1 - j\beta)x} + C\,e^{(\alpha_2 + j\beta)x} + D\,e^{(\alpha_2 - j\beta)x} + \frac{b_c\bar{m}' + d_c}{\omega} \tag{44}$$

where

$$\alpha_1 = \frac{1}{2\bar{\bar{a}}}(p+q), \quad \alpha_2 = \frac{1}{2\bar{\bar{a}}}(p-q), \quad \beta = \frac{r}{2\bar{\bar{a}}}$$

and

$$\begin{aligned} p &= -(\bar{b} + 2c\bar{m}') \\ q &= \sqrt[4]{(\bar{b} + 2c\bar{m}')^4 + (4\bar{\bar{a}}\omega)^2}\cos\sigma/2 \\ r &= \sqrt[4]{(\bar{b} + 2c\bar{m}')^4 + (4\bar{\bar{a}}\omega)^2}\sin\sigma/2 \\ \tan\sigma &= \frac{4\bar{\bar{a}}\omega}{(\bar{b} + 2c\bar{m}')^2} \end{aligned}$$

with

$$\bar{m}' \simeq \frac{\Delta M}{l} \tag{45}$$

and A, B, C, D are arbitrary constants to be determined by the boundary conditions.

Similarly,

$$m_c = -jA\,e^{(\alpha_1 + j\beta)x} + jB\,e^{(\alpha_1 - j\beta)x} + jC\,e^{(\alpha_2 + j\beta)x} - jD\,e^{(\alpha_2 - j\beta)x} - \frac{b_s\bar{m}' + d_s}{\omega}. \tag{46}$$

For the time-dependent part of the solution, the boundary conditions, equations (19) and (20), become

at $x = 0$

$$m_s = \delta M_0, \quad \text{and} \quad m_c = 0$$

and at $x = l$

$$m_s = \delta M_l\cos\phi, \quad \text{and} \quad m_c = \delta M_l\sin\phi.$$

These boundary conditions substituted in equations (44) and (46) above give simultaneous equations in A, B, C, D which expressed in matrix form are

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ e^{\lambda_1 l} & e^{\lambda_2 l} & e^{\lambda_3 l} & e^{\lambda_4 l} \\ -e^{\lambda_1 l} & e^{\lambda_2 l} & e^{\lambda_3 l} & -e^{\lambda_4 l} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} W \\ -jX \\ Y \\ -jZ \end{bmatrix}$$

where

$$\begin{bmatrix} W \\ -jX \\ Y \\ -jZ \end{bmatrix} = \begin{bmatrix} \delta M_0 - \left(\frac{b_c\bar{m}' + d_c}{\omega}\right) \\ -j\left(\frac{b_s\bar{m}' + d_s}{\omega}\right) \\ \delta M_l\cos\phi - \left(\frac{b_c\bar{m}' + d_c}{\omega}\right) \\ -j\left(\delta M_l\sin\phi + \left(\frac{b_s\bar{m}' + d_s}{\omega}\right)\right) \end{bmatrix} \tag{47}$$

and

$$\lambda_1 = \alpha_1 + j\beta, \quad \lambda_2 = \alpha_1 - j\beta, \quad \lambda_3 = \alpha_2 + j\beta, \quad \lambda_4 = \alpha_2 - j\beta$$

and m' is taken as $\Delta M/l$.

Upon solving for A, B, C and D the final solutions for m_s and m_c are

$$
\begin{aligned}
m_s = \frac{1}{\Delta}\{ & e^{\alpha_1 x}\cos\beta x(W(e^{(\alpha_1+\alpha_2)l}\cos 2\beta l - e^{2\alpha_2 l}) \\
& + X e^{(\alpha_1+\alpha_2)l}\sin 2\beta l - Y(e^{\alpha_1 l} - e^{\alpha_2 l})\cos\beta l \\
& - Z(e^{\alpha_1 l} + e^{\alpha_2 l})\sin\beta l) + e^{\alpha_2 x}\cos\beta x(W(e^{(\alpha_1+\alpha_2)l}\cos 2\beta l - e^{2\alpha_1 l}) \\
& - X e^{(\alpha_1+\alpha_2)l}\sin 2\beta l + Y(e^{\alpha_1 l} - e^{\alpha_2 l})\cos\beta l \\
& + Z(e^{\alpha_1 l} + e^{\alpha_2 l})\sin\beta l) + e^{\alpha_1 x}\sin\beta x(-X(e^{\alpha_1+\alpha_2)l}\cos 2\beta l - e^{2\alpha_2 l}) \\
& + W e^{(\alpha_1+\alpha_2)l}\sin 2\beta l + Z(e^{\alpha_1 l} - e^{\alpha_2 l})\cos\beta l \\
& - Y(e^{\alpha_1 l} + e^{\alpha_2 l})\sin\beta l) + e^{\alpha_2 x}\sin\beta x\,(X(e^{(\alpha_1+\alpha_2)l}\cos 2\beta l - e^{2\alpha_1 l}) \\
& + W e^{(\alpha_1+\alpha_2)l}\sin 2\beta l + Z(e^{\alpha_1 l} - e^{\alpha_2 l})\cos\beta l \\
& - Y(e^{\alpha_1 l} + e^{\alpha_2 l})\sin\beta l)\} + \frac{b_c\bar{m}' + d_c}{\omega}
\end{aligned}
\tag{48}
$$

where

$$\Delta = 2\,e^{(\alpha_1+\alpha_2)l}\cos 2\beta l - e^{2\alpha_1 l} - e^{2\alpha_2 l}.$$

Similarly,

$$
\begin{aligned}
m_c = \frac{1}{\Delta}\{ & e^{\alpha_1 x}\cos\beta x(X(e^{(\alpha_1+\alpha_2)l}\cos 2\beta l - e^{2\alpha_2 l}) \\
& - W(e^{(\alpha_1+\alpha_2)l}\sin 2\beta l - Z(e^{\alpha_1 l} - e^{\alpha_2 l})\cos\beta l \\
& + Y(e^{\alpha_1 l} + e^{\alpha_2 l})\sin\beta l) + e^{\alpha_2 x}\cos\beta x(X(e^{(\alpha_1+\alpha_2)l}\cos 2\beta l - e^{2\alpha_1 l}) \\
& + W e^{(\alpha_1+\alpha_2)l}\sin 2\beta l + Z(e^{\alpha_1 l} - e^{\alpha_2 l})\cos\beta l \\
& - Y(e^{\alpha_1 l} + e^{\alpha_2 l})\sin\beta l) + e^{\alpha_1 x}\sin\beta x\,(W(e^{\alpha_1+\alpha_2)l}\cos 2\beta l - e^{2\alpha_2 l}) \\
& + X e^{(\alpha_1+\alpha_2)l}\sin 2\beta l - Y(e^{\alpha_1 l} - e^{\alpha_2 l})\cos\beta l \\
& - Z(e^{\alpha_1 l} + e^{\alpha_2 l})\sin\beta l) + e^{\alpha_2 x}\sin\beta x\,(-W(e^{(\alpha_1+\alpha_2)l}\cos 2\beta l - e^{2\alpha_1 l}) \\
& + X e^{(\alpha_1+\alpha_2)l}\sin 2\beta l - Y(e^{\alpha_1 l} - e^{\alpha_2 l})\cos\beta l \\
& - Z(e^{\alpha_1 l} + e^{\alpha_2 l})\sin\beta l)\} - \left(\frac{b_s\bar{m}' + d_s}{\omega}\right)
\end{aligned}
\tag{49}
$$

SUMMARY

In summary the approximate solution to equation (22) under the boundary conditions (19) and (20) and temperature field (21) is given by

$$m(x, t) = \bar{M}_0 + \mu x + \nu x^2 + m_s(x)\sin\omega t + m_c(x)\cos\omega t$$

where μ is given by the appropriate root of equation (42), ν is given by equation (41), and m_s and m_c are given by equations (48) and (49) respectively.