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Efficient, Steady State Solution of a Time Variable RC Network, for Building Thermal Analysis

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In buildings, variable influences such as time dependent ventilation lead to thermal systems where the time-constant of the building becomes a function of time. This paper introduces an efficient method for obtaining the steady state, periodic thermal response of a building with arbitrary, time dependent ventilation. The method is applicable to a single time-constant thermal model but can be extended to higher order models. A simple, elementary numerical method for integrating the governing differential equation is proposed which does not need Fourier analysis, convolution or even evaluating an exponential, as required by most other methods. The initial value is also obtained explicitly. Hence, the usual initial period of integration—to get rid of transients—is not required. By direct comparison of the numerical method with an exact analytical solution in a special case, it is proved that the method is sufficiently accurate, provided the sampling interval is not too large compared to the thermal time-constant of the building. The method is further demonstrated by calculating the interior temperature of a building subjected to forced night cooling.

NOMENCLATURE

- ach ventilation rate [air changes per hour]
- C heat storage capacitance of massive structures [kJ/K]
- c_p specific heat [kJ/kg · K]
- f forcing function [kW]
- $H(t, \omega)$ transfer function
- Q_c convective load [kW]
- Q_r radiative load [kW]
- q heat energy stored in the massive structure [kJ]
- R_a mean film resistance from interior surface of shell to interior air [K/kW]
- R_o conductive shell resistance [K/kW]
- R_v equivalent ventilation resistance [K/kW]
- T diurnal period of 24 hours [h]
- T_c mean structure temperature [°C]
- T_{cr} required structure temperature for comfort [°C]
- T_{ir} required interior comfort temperature [°C]
- T_i zone interior air temperature [°C]
- T_{sa} effective sol-air external temperature [°C]
- T_o temperature of ventilating air [°C]
- T_x, T_y effective forcing temperatures [°C]
- T_t thermostat temperature [°C]
- ΔT time interval between sampling points [h]
- Vol zone volume [m^3]
- β coefficient of differential equation, equal to inverse of time-constant [/h]
- Γ anti-derivative of β (t)
- ρ specific density [kg/m^3]
- τ thermal time-constant of building [h].

Subscripts

- T interior temperature
- E energy loads
- S active systems.

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1. INTRODUCTION

THE ANALOGY between thermo-flow and electron-flow is often exploited to derive simple models for building thermal analysis [1–5]. In these models it is usual to lump the distributed thermal conductance and capacitance so that instead of the partial differential equation for heat conduction, one is faced with an ordinary differential equation, the order of which is determined by the number of lumped capacitances (nodes) used to describe the distributed parameters. Other heat transfer phenomena, such as radiative exchange and convection, can be integrated in the electrical analogy by defining the appropriate thermal sources and resistances. In the case of natural ventilation, the ventilation resistance is time dependent since ventilation rates vary appreciably with the hour of the day. An increase in the ventilation rate dramatically lowers the time-constant of the building as the interior air, and also the interior surfaces, come into contact with exterior air. To model radiative exchange requires a thermal resistance which is strongly dependent on the temperature, hence strongly non-linear [6] and therefore also time dependent. However, in most methods it is assumed that the parameters (resistances and capacitances) are time invariant, to enable the application of Fourier or Laplace techniques to obtain the solution. This limits the solution to cases where the ventilation rate is constant; a serious practical limitation.

In this paper we describe an elementary numerical method for solving the variable coefficient differential equation of the model of Mathews and Richards [1], without sacrificing functional accuracy or computation speed. Furthermore, the method is in no way restrictive regarding the magnitude or details of the variation of the parameters with time. The solution obtained is also stable and free of transients, even with discontinuous variations of the parameters. The method is generally applicable to the simple RC networks which many investigators use to model heat flow in buildings and other structures [2–5] by treating the higher order equations as a system of first order equations.

This paper commences with a discussion of some popular methods for thermal analysis in Section 2. In Section 3, the general governing equation for the thermal network from [1], with time dependent parameters, is given. In Section 4 some techniques for solving variable RC networks are discussed. A new efficient solution method is presented in Section 5 and the accuracy of this method is discussed in Section 6. In Section 7, it is shown how the model of [1] can be extended to include active systems, with indoor temperature controlled by a proportionally controlled thermostat. In Section 8 the extension of the method to higher order models is presented. Finally, Section 9 concludes with an example of the application of the method to night cooling of buildings.

2. METHODS FOR BUILDING THERMAL ANALYSIS

Before we discuss the method of [1] a brief discussion of the more popular methods for thermal analysis is in order.

A large number of computer programs for building thermal analysis are available. according to Tuddenham [7] there are more than one hundred in the United Kingdom and many hundreds elsewhere. Most of these methods are based on the 'admittance method' of the CIBS [8] or the 'response factor method' of ASHRAE [9]. The CIBS method was originally developed for manual calculations [10]. It employs pre-calculated tables of decrement and other factors for building materials. The response factor method was originally developed for computer implementation [11]. It does seem, however, that the method was severely influenced by the very crude computer hardware and software available at the time and the central theme of the method appears to be an attempt to ease the evaluation of the convolution integral, as required to obtain the forced response [6]. In view of the undreamt of growth in computer technology and numerical techniques in the last two decades, both these models appear outdated. The Fast Fourier Transform has made the evaluation of convolution integrals in the frequency domain a computationally efficient exercise.

In fact, it is today perfectly feasible, with the aid of two-port theory and cascade-matrices, to solve the partial differential equation for heat conduction exactly; as suggested by Athienitis [12, 13]. This approach is extremely powerful and very comprehensive thermal models may be used. However, it is limited to steady periodic solutions, which is—in our opinion—not a very serious drawback. Simulation of non-steady conditions seems rather academic. In practice one is confronted with continuous variables which can be described as quasi-periodic. One should therefore rather attempt to find the quasi-periodic solutions which are often sufficiently accurately represented by periodic solutions [12]. Especially if the principle of a design day, week or year is adhered to.

We believe there is scope for new methods. An important point is that new thermal analysis methods can either attempt to be extremely accurate, with emphasis on exact simulation—and be of academic value only, or sufficiently accurate with a design philosophy (e.g. design day) in mind—and practical. The highly refined models are more suitable for laboratory investigations of thermo-flow in buildings, with experimental verification under carefully controlled conditions. For design purposes, a simple thermo-flow model with a very clear physical interpretation and ready solution is more appropriate. The architect must be able to establish the thermal merits of a particular design in a convenient and efficient manner. The engineer needs a tractable model of the passive behaviour of the building to optimize his design. Architect and engineer need a clear physical interpretation of the model in order to rectify problems.

The required accuracy of a design tool needs careful consideration. In the initial stages of design many essential details, such as furnishings, are unknown. In fact, even if the design is complete in all details, it is doubtful that the thermal properties of all the materials are that well known, or that the influence of various construction methodologies on the thermal characteristics can be predicted. These uncertainties, combined with the vagaries of climate predictions and the behaviour of occupants and administrators, strictly limit the practical attainable accuracy. In our opinion, thermal performance tools must attempt to model the essential thermal characteristics of buildings. The first task of the designer is to ensure the basic thermal properties of the building, such as the shell resistance and the thermal time constant.

It was indicated by Mathews and Richards [1] that a simple, single time-constant RC network, as given in Fig. 1, can be a very useful aid for determining the thermal performance of a building. This extraordinary simple approach has a number of important advantages; the thermal response analysis can be implemented on inexpensive, generally available computers (PCs) and still provides fairly accurate answers, while requiring a minimum amount of effort and time. In addition, the simple thermal network has a very clear physical interpretation, which is easily explained to designers. The simplicity also enables easy extension to include new ideas and building materials. This facilitates a design process where, once the basic design has been described, various options can be evaluated in a couple of seconds. The data input

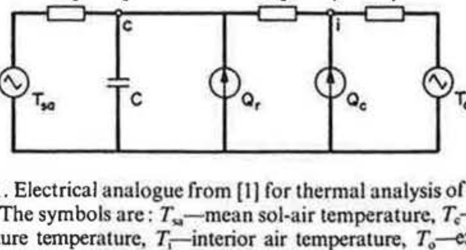

Fig. 1. Electrical analogue from [1] for thermal analysis of buildings. The symbols are: T_{sa}—mean sol-air temperature, T_c—bulk structure temperature, T_i—interior air temperature, T_o—exterior air temperature, R_o—shell thermal resistance, R_a—mean interior surface film resistance, R_v—effective ventilation resistance. C is the active heat storage capacitance of the structure [14].

requirements are modest since only the essential features of the building are described. No expert knowledge of thermal analysis is required.

The main limitations of the method are the single zone approach, the assumption of isothermal interior surface temperature and the assumption of well mixed interior air. The validity of these assumptions is difficult to establish, but nevertheless, the method is vindicated by the excellent results obtained from verification experiments in many buildings [1]. Also, in our opinion, the number of practical cases requiring detailed multi-zone thermal analysis is very small.

The details of the method appear in [1]. Briefly, in Fig. 1 the resistances are: R_o—conductive shell resistance including exterior film coefficients, R_a—film resistance of the interior surfaces and R_v—ventilation resistance. The ventilation resistance is obtained from $R_v = 3.6/Vol \cdot \rho \cdot ach \cdot c_p$ where Vol [m^3] is the interior volume, ρ [kg/m^3] is the density, c_p [kJ/kg $\cdot$ K] is the specific heat of air and *ach* is the air change rate per hour. The sources in Fig. 1 are: T_{sa}—averaged sol-air temperature of the external surfaces, Q_r—mean radiation on the interior surfaces, Q_v—interior air convective sources and T_o—temperature of the ventilating air. The circuit can be easily extended to include also structural cooling, evaporative cooling etc. by adding more sources. The heat storage of the massive elements of the construction is represented by the capacitor C. The value of C, which is critical, includes only the active part of the total heat capacitance of the zone and is determined in a heuristic, but theoretically well founded and experimentally well proven method [14]. Typical values for the network elements for a selection of building zones are given in Table 1. Some relevant details of the zones are supplied in Table 2.

The dependent quantities of interest are firstly: the interior air temperature T_i, and secondly: the sensible load required to maintain a specified interior temperature. (Latent loads present no problem, but for simplicity's sake this paper will be restricted to dry-bulb temperatures. A new version of the program is commercially available which includes evaporative cooling, latent loads and structural cooling.) The method has been extensively validated [1, 14] and has already, in practice, proved a valuable design aid for architects, and for establishing norms for the design of thermally efficient new buildings. The implementation is restricted to periodic, diurnal forcing functions. The philosophy is to calculate the response for a typical hot and cold design day as though every other day is exactly similar. In practice, the method will be grossly erroneous only when the building has a very long time constant (very massive with high shell isolation) and the thermal energy consumption differs radically on some days from the norm e.g. on weekends, or if the weather pattern is drastically different on a single day. This approach is in line with the philosophy of a design day, where days of typical extreme weather are used. There is no fundamental reason why the method is not applicable to periods of e.g. 7 or 365 days.

Previously the solution of the network was obtained by convolving the forcing functions and system response in the frequency domain [1]. This method assumes the parameters R_o, R_a, R_v and C are constants in time and the governing equation of the network is a linear, constant coefficient, differential equation. This is rather restrictive; with this assumption the program cannot treat situations where e.g. the ventilation rate (and therefore R_v) varies with the hour of the day. Consequently, the method is only applicable to situations where the windows remain closed or open during all hours and/or a constant rate of forced ventilation and infiltration is maintained throughout the day. Most other thermal analysis programs suffer from the same limitation. To relax this restriction, it is necessary to solve the governing equation of the network of Fig. 1, with circuit elements which are assumed functions of time; a considerably more difficult problem.

3. GOVERNING DIFFERENTIAL EQUATION

The equation for the interior air temperature of the network of Fig. 1, *with resistors and capacitance assumed functions of time*, is

$$T_i = \frac{T_c \cdot R_v + T_x \cdot R_a}{R_v + R_a} \tag{1}$$

where:

$$T_x = T_o + R_v \cdot Q_c. \tag{2}$$

T_c is the temperature at the structure node (across the capacitor in Fig. 1) and is given in terms of the stored heat q

$$T_c = q/C \tag{3}$$

which is found from the governing differential equation

$$\dot{q} + \beta_T \cdot q = f_T. \tag{4}$$

Table 1. Numerical values of circuit parameters for some typical building zones. The time-constant for interior temperature, and also the time-constant for load calculation are given for ventilation rates of 0.1 and 30 *ach*

	Thermal parameter(s)			*ach* = 0.1		*ach* = 30		
Building	C [kJ/K]	R_o [K/kW]	R_a [K/kW]	R_v [K/kW]	τ_T [h]	R_v [K/kW]	τ_T [h]	τ_E [h]
Shed	416521.70	0.0656	0.0650	9.934	7.54	0.0331	4.55	3.78
Hut	1822.77	33.8771	2.0330	1333.333	16.73	4.4444	2.75	0.97
Factory	1593885.15	0.0422	0.0060	0.335	16.62	0.0011	2.70	2.33
Room	3968.47	29.0070	2.0740	1333.333	31.30	4.4444	5.87	2.13
Shop	45398.16	2.7359	0.3150	69.231	33.19	0.2308	5.74	3.56
Office	61780.04	8.3667	1.4180	878.049	142.23	2.9268	49.08	20.81

Table 2. Some construction data of the building zones of Table 1

Description of buildings	Floor area [m²]	Shell area [m²]	Volume [m³]	Window area [m²]	Roof	Floor	Walls
Shed	763	1290.8	3624	12.6	exposed steel	concrete on ground	steel
Hut	9	48.3	27	1.76	exposed steel airspace fibreboard	concrete on ground	double brick
Factory	7755	13292	107500	1335	exposed steel glass wool	concrete on ground	steel glass wool
Room	11	19.5	27	1.79	exposed steel airspace glass wool gypsum	carpet concrete on ground	brick cavity brick
Shop	102	185.7	520	24.92	exposed slate glass wool airspace fibreboard	PVC concrete suspended	double brick
Office	14	9.9	41	3.91	not exposed concrete	PVC concrete	brick cavity brick

In (4) the subscript T refers to the solution for the interior air temperature. The forcing function f_T is given by

$$f_T = \frac{T_x}{R_a + R_v} + \frac{T_y}{R_o} \tag{5}$$

with

$$T_y = T_{sa} + R_o \cdot Q_r$$

and the time dependent coefficient $\beta_T(t)$ is the inverse of the time-constant τ_T of the building

$$\beta_T = \frac{R_a + R_o + R_v}{C \cdot R_o \cdot (R_a + R_v)} = 1/\tau_T. \tag{6}$$

In (6) it is seen that the time-constant of the building varies between a maximum value of $C \cdot R_o$ with no ventilation ($R_v \to \infty$), and a minimum of $C \cdot R_a \cdot R_o/(R_a + R_o)$ when the ventilation rate is very large ($R_v \to 0$).

It is possible to substitute (2) to (6) in equation (1) to obtain an equation which directly delivers T_i in terms of the sources. In practice, it is a great advantage to write the governing equation in terms of the amount of stored heat, and not in terms of the primary quantities of interest. The stored heat is a fairly smooth function of time (provided mass is not added to or removed from the structure), while the temperatures are subject to sharp discontinuities when the values of the circuit elements suddenly change. Since we are especially interested in sudden, large changes, e.g. when windows are opened and closed or forced cooling is switched on and off, it is important to be able to solve the equation accurately for discontinuous coefficients.

Note that the equations were derived for the general case where any of the elements, and not just R_v, may be subject to variation although R_v is the most important. There is little formal distinction between changes in R_v and changes in the other elements. Hence the only type of variation the equations do not cater for is when the storage capacity of the building is varied by introducing or removing mass. Variation in the other parameters is of more than theoretical importance, e.g. the interior film resistance is definitely also affected by the time-dependent air circulation rate. However, since we currently have no reliable model for these influences they have been ignored in the rest of this paper, and constant values for R_a, R_o and C were used in the numerical calculations.

Analogous to equations (1) to (6), the sensible convective load Q_{cr}, to obtain a prescribed interior temperature T_{ir}, can be found by substituting T_{ir} for T_i in (1) to (6) and solving for the convective load. While this load calculation is highly theoretical—practical thermostats normally include dead bands and/or hysteresis—the calculation is useful for estimating required system capacitances, without troublesome iterative procedures. Q_{cr} is given by

$$Q_{cr} = \frac{R_a + R_v}{R_a \cdot R_v} \cdot T_{ir} - \frac{T_{cr}}{R_a} - \frac{T_o}{R_v} \tag{7}$$

with T_{cr} the required structure temperature. T_{cr} is determined from the amount of stored heat q_r

$$T_{cr} = q_r/C \tag{8}$$

which satisfies the differential equation

$$\dot{q}_r + \beta_E \cdot q_r = f_E. \tag{9}$$

In this case the forcing function f_E is given by

$$f_E = \frac{T_y}{R_o} + \frac{T_{ir}}{R_a} \tag{10}$$

and the coefficient β_E is

$$\beta_E = \frac{R_a + R_o}{R_a \cdot R_o \cdot C} = 1/\tau_E. \tag{11}$$

It is seen that the convective load time-constant τ_E corresponds with the air temperature time-constant when the ventilation rate is very large, that is with the minimum value of τ_T above. In (11) the ventilation resistance R_v plays no role, because the ventilation load is completely canceled by the convective system, so that the interior air-temperature is maintained at the set point. In both instances, (4) and (9), the solution of a linear first order differential equation with time dependent coefficient $\beta(t)$ must be found. The equation is of the form

$$\dot{y}(t)+\beta(t)\cdot y(t) = x(t). \qquad (12)$$

4. EXISTING METHODS OF SOLUTIONS

The general solution of (12) is well known and can be found in any text book on elementary differential equations. It is

$$y(t) = \exp[-\Gamma(t)]\cdot\int_{-\infty}^{t}\exp[\Gamma(t)]\cdot x(t)\,dt \qquad (13)$$

with

$$\Gamma(t) = \int\beta(t)\,dt. \qquad (14)$$

In (13) the lower limit of integration is taken at minus infinity to indicate that the steady state response is required. The integral can be evaluated by numerical integration (either in the form of (13), or in the form of the original differential equation (12)) by a standard procedure such as Runge–Kutta. Standard numerical integration is, unfortunately, relatively inefficient. The initial condition is unspecified, or stated more precisely: is assumed to have occurred far back in history. The integration must continue until the transient response is extinct. Since the time constant of a building can be quite long (50 hours or more is not uncommon), and the transient response can be regarded as sufficiently extinct only after 5 time constants, integration may have to continue for a considerable period to ensure sufficient accuracy of the answer. Since a very high premium is attached to the speed of computation, it is desirable to find a quicker method.

Various methods for treating systems with variable parameters exist in the literature. It was shown by Carson [15] (see also [16]) that the solution can be expressed in the form of a Volterra integral equation. The solution of this integral equation is given in terms of an infinite progression. Alternatively, solutions in terms of a series expansion of Bessel functions can be found when the coefficient varies sinusoidally [16]. These and other similar expansions [17, 18], converge rapidly when the variation of the coefficient $\beta(t)$ is slow compared to the variation of the forcing function, or when $|\dot{\beta}/\beta| \ll 1$. For sudden jumps in the value of β, i.e. when $\dot{\beta}$ is very large, they are of little practical value.

The traditional method for isolating the steady state response is through Fourier series methods. In essence there is little fundamental difference in the application of this method to systems with variable parameters—as opposed to constant parameters—except that it must be assumed that the Fourier coefficients of the output are functions of time [17]. The method leads to a mixed time-frequency domain description. The Fourier technique is so prevalent in the literature that it warrants some further discussion.

Since the differential equation is linear (although time variant) it will be sufficient to determine the solution for the phasor $x(t) = X\cdot e^{j\omega t}$ ($X = X(\omega)$ a complex number independent of time). The solution for general periodic inputs can be obtained by superpositioning the phasor components of each constituent frequency component. Assuming the response $y(t)$ to the phasor input $x(t)$ is of the form $y(t) = Y(t,\omega)\cdot e^{j\omega t}$ and substituting these assumed values for x and y in (12) furnishes:

$$\frac{\partial Y(t,\omega)}{\partial t}+[j\omega+\beta(t)]\cdot Y(t,\omega) = X(\omega). \qquad (15)$$

This is the modulation function equation (MFE) of the system as discussed in [18]. In [17] and [18] methods are presented for directly transforming (12) into (15) for more general systems. It is customary to define the system transfer function:

$$H(t,\omega) = \frac{Y(t,\omega)}{X} \qquad (16)$$

which, from (15), satisfies:

$$\frac{\partial H}{\partial t}+[j\omega+\beta]\cdot H = 1. \qquad (17)$$

Equation (17) is of exactly the same form as (12), however, it involves complex functions and the input in (17) is a constant. But obviously, the Fourier series expansion is not beneficial regarding computation time. Instead of the initial value, $y(0)$ in (12), the initial frequency response $H(0,\omega)$ is required, and furthermore, (17) must be solved for each frequency component of the forcing function. However, if the time-constant is sufficiently long at all hours, only the first few frequency components will be significant in the solution.

A straightforward and useful approximate solution to the time dependent problem is to ignore the derivative term $\partial H/\partial t$ in (17), yielding the approximate system transfer function $H_f(t,\omega) \approx 1/[j\omega+\beta(t)]$, which may be called the *frozen system function* after [17]. This type of approximation is frequently employed, e.g. when time dependent admittance factors or transfer functions are instantaneously varied in the solution procedure. Actually, the building cannot immediately adjust to a change in parameters, the adjustment is subject to the constraints imposed by (17). If H_f is substituted in (17) it is immediately seen that the approximation is very crude when sudden changes occur. The frozen system is a good approximation when the changes in the parameters take place over a period of time considerably longer than the current time-constant. The error incurred by the use of H_f for H may be interpreted in a physical way by noting that the calculated response will overreact to sudden changes, since it is assumed that the system can immediately adapt to changes.

Another approach is to assume the circuit parameters are constant in small intervals and then to solve the equation exactly for each interval [19]. This approach requires matching the final conditions of each interval to the initial conditions of the next interval. For a steady, periodic solution, the initial condition of the first interval must match the final condition of the last interval. It is seen that the process requires the simultaneous solution of a large number of equations. In the next section we obtain an approximate method following this approach with the additional assumption that all forcing functions and output variables are constant between sampling points, in which case the integration is trivial and an explicit formulation for the initial value is possible.

5. APPROXIMATE NUMERICAL SOLUTION

The advantage of writing the governing equation in terms of the amount of stored energy is; it reduces the sensitivity of the solution of T_i to errors in the solution of the differential equation (12). A large part of the variation in T_i is accurately included in the final calculation of T_i and Q_{ri} from q, in (1) and (7). This is easily demonstrated by noting that in the limit, when the ventilation rate is very large ($R_v \sim 0$), the sole contributor to T_i in equation (1) is T_x. On the other hand, when R_v approaches infinity, the sole contributor is T_c. This is in accordance with the network of Fig. 1.

To obtain a numerical solution for (12) we follow the standard procedure of rewriting the equation in the form of an integral equation

$$y(t) = \int_{-\infty}^{t} [x(t_1) - \beta(t_1)\cdot y(t_1)]\,dt_1$$

$$= y(0) + \int_{0}^{t} [x(t_1) - \beta(t_1)\cdot y(t_1)]\,dt_1. \quad (18)$$

For periodic, steady state solutions with period T, it is required that the initial value of every period equals the final value of the previous period

$$y(0) = y(T) = y(0) + \int_{0}^{T} [x(t_1) - \beta(t_1)\cdot y(t_1)]\,dt_1$$

and therefore

$$\int_{0}^{T} [x(t_1) - \beta(t_1)\cdot y(t_1)]\,dt_1 = 0. \quad (19)$$

The steady, periodic solution of (12) is given by (18) with boundary condition stipulated by (19). For discrete data at $t = t_i = i\cdot\Delta T$, $T = N\cdot\Delta T$, these equations take the form

$$y_k = y_0 + \sum_{i=0}^{k-1} \Delta T\cdot(x_i - \beta_i\cdot y_i) \quad (20)$$

and

$$\sum_{i=0}^{N-1} \Delta T\cdot(x_i - \beta_i\cdot y_i) = 0. \quad (21)$$

It is assumed all variables are constant between sampling points and $x_i = x(t_i)$ and $\beta_i = \beta(t_i)$ are tabulated functions. Equation (20) can be written in the following corresponding, iterative form

$$y_k = \Delta T\cdot x_{k-1} + (1 - \Delta T\cdot\beta_{k-1})\cdot y_{k-1} \quad (22)$$

$$k = 1, 2, 3, \ldots, N-1$$

If one value of y is known the other values are easily found from (22), provided the iteration is stable. A closed form solution for the initial value y_0 is obtained by starting with

$$y_N = \Delta T\cdot x_{N-1} + (1 - \Delta T\cdot\beta_{N-1})\cdot y_{N-1} = y_0 \quad (23)$$

and substituting previous values of y_k.

$$\begin{aligned} y_N &= \Delta T\cdot x_{N-1} + (1 - \Delta T\cdot\beta_{N-1})\cdot(\Delta T\cdot x_{N-2} \\ &\quad + (1 - \Delta T\cdot\beta_{N-2})\cdot(\Delta T\cdot x_{N-3} \\ &\quad + (\ldots\Delta T\cdot x_0)\ldots))) + (1 - \Delta T\cdot\beta_{N-1}) \\ &\quad \cdot(1 - \Delta T\cdot\beta_{N-2})\cdot\ldots\cdot(1 - \Delta T\cdot\beta_0)\cdot y_0 = y_0. \end{aligned}$$

This can be rewritten compactly

$$y_0 = \frac{\left[\sum_{k=0}^{N-2} \Delta T\cdot x_k\cdot \prod_{j=k+1}^{N-1} (1 - \Delta T\cdot\beta_k)\right] + \Delta T\cdot x_{N-1}}{1 - \prod_{k=0}^{N-1} (1 - \Delta T\cdot\beta_k)}. \quad (24)$$

When programming equation (24), advantage can be taken of the fact that the product expression occurs both in the numerator and the denominator, by starting with the highest value of k and counting down instead of up. By storing at each step k the partial sum and the partial product, the product expression can be evaluated successively for every term in the sum. The product expression in the denominator is found by multiplying the final product of the numerator with $(1 - \Delta T\cdot\beta_0)$.

The complete approximate solution is given by (22) with initial value given explicitly by (24). In effect the iteration is carried out twice through one period. The first iteration is used to determine the initial value, and the second calculates the results at all hours. We maintain that the method is efficient; the initial value is determined after a single iteration, instead of after iterating through five time-constants. It will be shown in the next section that sufficient accuracy can be obtained from a fixed step size.

Firstly we must establish the stability of (22). To investigate the stability of (22) the $\mathscr{Z}$ transform method [20] will be used under the assumption that β is independent of time. The transfer function of (22) in the z domain is then

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}\cdot\Delta T\cdot\beta}{1 - z^{-1}\cdot(1 - \Delta T\cdot\beta)}. \quad (25)$$

The time invariant system is stable if the pole of the transfer function lies inside the unit circle on the complex z-plane, i.e.

$$|1 - \Delta T\cdot\beta| \leqslant 1. \quad (26)$$

It must be noted that this condition will not guarantee stability in the time variable case, but it is indicative of stability [21]. In practice we gave found the iteration stable under the condition (26) even for sudden large changes in the value of β. According to (26) the sampling interval is chosen so that $\Delta T\cdot\beta_m < 2$, where β_m is the maximum value attained by $\beta(t)$. For most practical buildings a sampling period of 1 hour suffices. For extremely light constructions we have found it necessary to decrease the sampling rate to 15 min. An equation very similar to (24) can be obtained by using a backward difference in (22). It has the advantage of unconditional stability. However, we have found the solution given by equations (22) and (24) quite satisfactory in practice. The complete solution for interior air temperature is given by equations (1) to (6), with the solution of (6) obtained via (22) and (24), with q, f_T and β_T replacing y, x and β respectively. Similarly, the convective load required to maintain a prescribed interior air temperature is given by (7) to (11), where the solution of equation (9) is again given by (22) and (24), with q_r, f_E and β_E now replacing y, x and β.

6. ACCURACY OF THE METHOD

The method of Section 5 is essentially Euler's method for the numerical integration of a differential equation. An upper bound for the total propagated error is [22]

$$|e| \leqslant \frac{\Delta T \cdot |\ddot{y}|}{2 \cdot \beta} \cdot [\exp(\beta \cdot T) - 1]. \tag{27}$$

The error grows rapidly when $\beta \cdot \Delta T = \Delta T/\tau > 0.1$. $|\ddot{y}|$ may be quite large for sudden changes in β. In fact, the derivative of (12) gives

$$\ddot{y} = \dot{x} - \beta \cdot x(\beta^2 - \dot{\beta}) \cdot y. \tag{28}$$

$|\ddot{y}|$ contains a term proportional to $\dot{\beta}$ which might be large for large variations of $\beta(t)$. Actually, the numerical integration technique is rigorously exact if all variables assume constant values between sampling points, even with discontinuous derivatives at the sampling points. Equation (27) must not be taken too seriously, it is derived under the assumption of continuous functions. To decrease error propagation and accuracy for continuous signals, one can use a higher order numerical approximation technique. Beginning with the trapezoid rule, for instance, an exactly similar scheme with local error theoretically proportional to the third derivative and square step size results. These and a host of other higher order approximation techniques [23] are not as advantageous for discontinuous input functions. Unless the integration is done over continuous subintervals, they tend to smooth the discontinuities and to make the solution appear non-causal, since they pre-empt the sudden change. The effect is easily explained by noting that the higher order techniques in effect interpolate between the sampling points so that values in the immediate future will influence the present result. In practice, we have found the Euler algorithm sufficiently accurate and quick.

To obtain a practical evaluation of the accuracy of the method, the approximate solution can be compared with the exact solution in a special case. We take the case where $\beta(t)$ is constant everywhere, except at two points where the value jumps discontinuously i.e. $\beta(t)$ given by

$$\beta(t) = \begin{cases} \beta_0 & \text{when} \quad 0 \leqslant t < T_1 \\ \beta_1 & \qquad\quad T_1 \leqslant t < T \end{cases}$$

The exact solution for a constant β and a sinusoidal input function given by $x = X \cdot (1 + m \cdot \cos \omega[t + t_o])$ is from the Laplace transform

$$y(t) = y(0) \cdot e^{-\beta t} + A(t) \tag{29}$$

where

$$A(t) = \frac{X}{\beta} \cdot \left[1 - e^{-\beta t} + \frac{m\beta}{\sqrt{\beta^2 + \omega^2}} \cdot \alpha(t) \right]$$

$$\alpha(t) = \cos(\omega[t + t_o] - \varphi) - \cos(\omega t_o - \varphi) \cdot e^{-\beta t}$$

and $y(0)$ is the initial value, $\tan \varphi = \omega/\beta$. Next, apply this solution to the intervals in (28) and set $y(0) = y(T)$, and $y(T_1)$ continuous. The solution for $y(0) = y_o$ is

$$y_0 = \frac{A_1 + A_0 \cdot e^{-\beta_1(T - T_1)}}{1 - e^{-\beta_0 T_1 - \beta_1(T - T_1)}} \tag{30}$$

with

$$A_0 = A_o(T_1) \quad \text{and} \quad A_1 = A_1(T - T_1).$$

The subscripts 0 and 1 of A and β in (30) refer to the first and second intervals respectively. Figure 2 shows the error between analytic solution (29), (30), and approximate numerical solution (24), (22), for a building with relatively short time-constant. The building (an agricultural shed) has a time-constant of 7.5 hours (see τ_T (Table 1)) with closed windows. This is a very short thermal time-constant and a practical sampling rate would be 15 min, but to show the robustness of the method, a sampling period of 1 h is used in the calculation. The ventilation rate jumps from 0.1 to 30 *ach* resulting in a time-constant jump from 7.5 to 4.6 h, the jump occurring at $T_1 = 11$ h. The forcing functions used for the calculation are:

$$T_{sa} = 20 + 10 \cdot \cos(2\pi/24 \cdot t)^\circ\text{C},$$

$$T_o = 20 + 5 \cdot \cos(2\pi/24 \cdot t)^\circ\text{C} \quad \text{and}$$

$$Q_c = Q_r = 0 \text{ kW}.$$

The figure shows the error obtained by a sudden increase in the number of air changes as well as a sudden decrease of similar strength. In this worst case, $\beta \cdot \Delta T = 1/4.6$, the temperature error is less than 1°C. The error is decreased

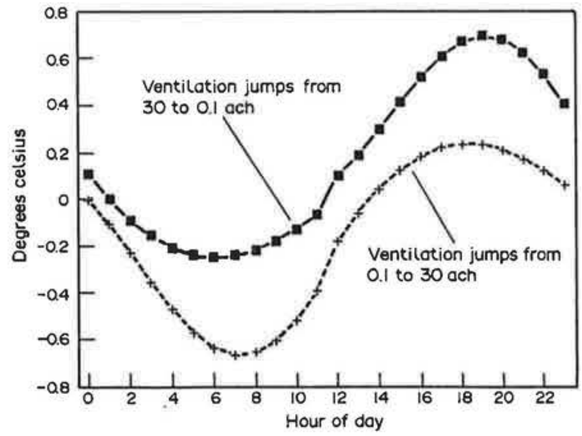

Fig. 2. Difference between analytically derived exact prediction of interior temperature and numerical algorithm for a sudden jump in ventilation rate and sinusoidal forcing functions. The sampling period is 1 h. Building: agricultural shed (see Table 2), time-constant $\tau_T = 4.6$ h (see Table 1), with ventilation rate 30 *ach*. The upper and lower traces show the error when the ventilation-rate jumps from 0.1 to 30 *ach*, and from 30 to 0.1 *ach* respectively. The jump occurring at the 11th hour.

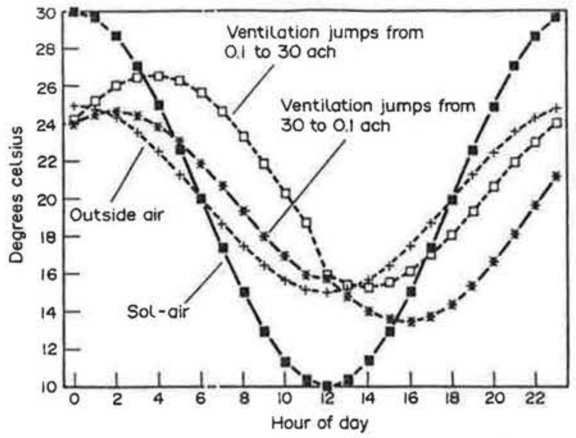

Fig. 3. Predicted interior temperature for the low-mass building of Fig. 2 when the ventilation-rate jumps from 0.1 to 30 *ach*. The assumed sol-air, outside air, and two predicted interior air temperatures are shown, for the cases where the ventilation-rate jumps from 0.1 to 30 *ach*, and from 30 to 0.1 *ach* respectively.

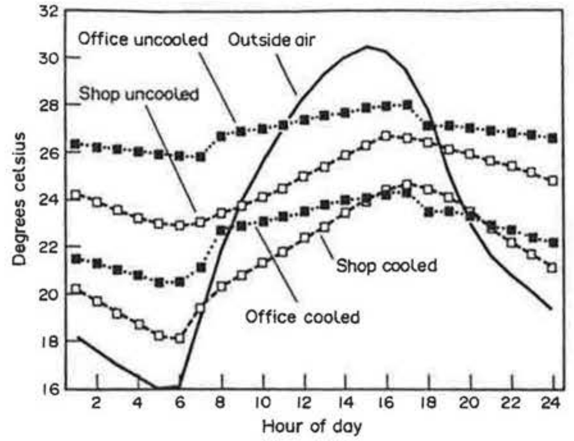

Fig. 4. Night cooling of a shop and a massive office. The traces marked 'uncooled' are with a ventilation-rate of 1 *ach* during the hours 1700 to 0900, and 2 *ach* during the hours 0900 to 1700. The other traces show the effect of increasing the ventilation to 20 *ach* during the hours 2000 to 0600. The internal load during the hours 0800 to 1700 consists of 0.5 kW and 2 persons.

to insignificant levels by decreasing the sampling period to 15 min, with linear interpolation between sampling points. Figure 3 shows the resulting interior temperatures. Note the sharp discontinuity. In practice, the heat capacitance and the finite mixing time of the interior air (both neglected in the model) will tend to smooth the discontinuity so that a smooth transition will be measured. (We have also found that with sudden changes in the ventilation rate the time constant of the thermograph can often not be neglected.)

The calculations were repeated for a building with a longer time-constant (office block), where the time constant jumped from 142 to 49 h when the ventilation rate jumped from 0.1 to 30 *ach*. The error between the analytic and approximate solutions in this case, $T = 1$ h, $\beta \cdot \Delta T = 1/49$, was less than 0.1°C.

7. PROPORTIONAL FEEDBACK, ACTIVE SYSTEMS

The method is very easy to extend to active indoor convective systems. If another system convective load, Q_s, given by

$$Q_s = \alpha \cdot (T_i - T_t) \quad (31)$$

with α the proportional feedback gain [kW/K] and T_t the thermostat temperature, is added to the convective load Q_c in (2), it is found that the behaviour of the system is again governed by (4) but with parameter $\beta(t)$ now given by

$$\beta_S(t) = \beta_T - \frac{\alpha \cdot R_v^2}{C \cdot (R_a + R_v) \cdot (R_a + R_v - \alpha \cdot R_a R_v)}. \quad (32)$$

In the limit $\alpha \to \infty$, $\beta_S \to \beta_E$ in accordance with the root-locus theorem for closed loop systems. It is seen that by this simple re-definition of β the method can be extended to proportionally controlled systems. Practical thermostats include non-linearities such as dead bands and hysteresis. These effects can also be included in the model, but the solution of the model becomes arduous; the initial value must be found by successive approximation [23].

8. DEMONSTRATION OF THE METHOD

To conclude the paper, we demonstrate the application of the method to predicting the interior temperature in buildings subjected to night-time forced ventilation. Figure 4 gives the interior temperature obtained for a building of medium (shop) and long time-constant (office); with a ventilation rate of 1 *ach* during the hours 1700 to 0900 and a daytime ventilation rate of 2 *ach*, and secondly, with a ventilation rate of 20 *ach* from 2000 to 0700 and 2 *ach* during the rest of the day. The outside air temperature is also shown. Both buildings are modelled with an interior load of 2 persons and 0.5 kW during office hours. Both buildings respond favourably. The peak temperatures drop nearly 4°C in the office and 2°C in the shop, bringing them close to the comfort range. It may be noted that these results are in qualitative agreement with the conclusions reached in [24], obtained with a much more sophisticated finite difference method. The authors will attempt to verify predictions like these in the near future by actual measurement in test huts and real buildings.

9. CONCLUSION

The method of Section 4 has been implemented in a Pascal routine which calculates the solution of the variable network in a fraction of a second on an ordinary PC. The method is simple, straightforward, efficient and sufficiently accurate. It does not require Fourier analysis or even the evaluation of exponentials.

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