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The Scaling of Flows of Energy and Mass Through Stairwells

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Simple physical and dimensional arguments are used to determine the laws relating the mass flow induced within a closed recirculating system representative of a domestic stairwell to the energy input driving the flow and the temperature differential established between chambers above and below the stairwell. Appropriate dimensionless groupings are introduced to characterise this kind of system. Experimental results obtained from a one-half scale model of a stairwell are used to validate the simplest analysis and to define the influence of a Reynolds number characteristic of the flow. The results are used to investigate the utility of models of various scales.

1. INTRODUCTION

IN RECENT years much effort has been devoted to the development of energy models of complete buildings. Typically, these models establish energy balances for each of a number of 'zones' into which the building can be divided and then solve the numerical problem posed by these simultaneous constraints, to determine a consistent pattern of transfers between the zones. The inputs to such a model are solar radiation to or through the envelope of the building and heat release within the several zones, together with a specification of any forced ventilation which may be provided. It is also necessary to define in a realistic manner the mechanisms of transfer between the 'zones', typically discrete rooms or other spaces within the building. The present paper deals with one of the most important mechanisms of inter-zone transfer, namely, the flows up and down the stairwells that connect the individual storeys of buildings. For definiteness, our study has concentrated upon stairwells typical of those occurring in houses, although the principles are more widely applicable.

As part of the 'Energy in Buildings' Specially Supported Programme of the Science and Engineering Research Council a number of linked investigations have been carried out, using physical models (at one-half and one-tenth scales) to investigate the detailed flow processes and the overall performance of various stairwell geometries, and using computational models to investigate features of the flows not readily determined using the few dozen measuring probes that can be deployed in a physical model. These aspects of our work will be described in other papers. The immediate concern of this paper is the development of the simplest kind of analytical modelling of the flow processes within stairwells. This provides the means of designing experiments on stairwells and of interpreting their results. It also suggests the structure of the formulae that should be introduced into a computer model of energy transfer within a building to characterise in a more realistic manner the flows of mass and energy between the several floors of the building.

In view of the evident importance of transfers through a stairwell for the energy balances within a typical dwelling, it is surprising that so little information exists in the literature on these processes. The only detailed study known to us is a recent paper by Feustel *et al.* [1], and that deals with the stack effect in a multi-storey building and is therefore not relevant to the more typical situation in domestic accommodation a few storeys in height.

2. FUNDAMENTAL DIMENSIONAL CONSIDERATIONS

Figure 1 illustrates the essential features of the system to be considered; only a closed, recirculating system is taken into consideration in this first analysis. Conditions will be taken to be essentially constant in time, though this class of flows must be expected to exhibit random fluctuations about time-mean values. A flow passage whose significant cross-sectional area is A connects an upper and a lower chamber, where the mean temperatures are T_2 and T_1, respectively. The effective difference in elevation of the two chambers is h, and an air flow at rate $\dot{V}$ circulates between the chambers, driven by a supply of energy at rate $\dot{Q}$ to the lower chamber, the balancing extraction of energy (by conduction to the walls) being $\dot{Q}_1$ for the lower chamber and $\dot{Q}_2$ for the

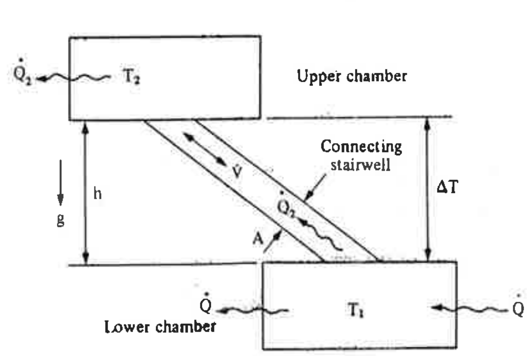

Fig. 1.

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upper. The fluid-mechanical resistance to the upwards flow $\dot{V}$ and the equal compensating downflow is indicated by the dimensionless loss coefficient K, which must be expected to depend upon the Reynolds number Re of the system, so that $K = f(\text{Re})$, where Re is some Reynolds number characterising the process.

In Fig. 1 an elongated connection between the upper and lower storeys is shown. In realistic stairway configurations the upper and lower volumes merge more abruptly; hence it is realistic to neglect any possible heat transfer from the stairwell itself.

Certain properties of the circulating fluid also play a part, namely: the bulk coefficient of expansion, $\beta = 1/T$ (where T is the mean absolute temperature of the system), ρ, the mean density, and c_p, the specific heat. Throughout this discussion, we shall assume that the temperature ranges only over a few degrees, so that an intermediate value can be adopted without significant error, in calculating β, ρ and $\dot{V}$.

In terms of these parameters, we can postulate that the circulation established by the supply of heat at the lower level is dependent upon the other quantities as follows:

$$\dot{V} \sim \dot{Q}, \dot{Q}_2/\dot{Q}_1, A, K, h, g, \beta, \rho, c_p. \tag{1}$$

However, in the basic force balance for the system, the combination βgh arises, and it is only in this way that these quantities enter into the specification of the motion. Also, in the energy balance the combination $\dot{Q}/\rho c_p$ occurs. Hence the relations of the general class (1) that are of physical significance are of the form:

$$\dot{V} \sim \dot{Q}/\rho c_p, \beta gh, A, \dot{Q}_2/\dot{Q}_1, K. \tag{2}$$

The four dimensioned quantities appearing here contain three independent dimensions. As a consequence the result can be compressed by the introduction of a single additional dimensionless grouping:

$$\frac{(\dot{V}/A)^3}{(\dot{Q}\beta gh/\rho c_p A)} = f(\dot{Q}_2/\dot{Q}_1, K) \quad \text{or} \quad f(\dot{Q}_2/\dot{Q}_1, \text{Re}). \tag{3}$$

Introducing a Froude number to characterise the force balance

$$\text{Fr} = \frac{\dot{V}}{A(gh)^{1/2}} \tag{4}$$

and a Stanton number to characterise the energy balance

$$\text{St} = \frac{\dot{Q}}{\rho c_p T A (gh)^{1/2}} \tag{5}$$

we obtain

$$\text{Fr}^3/\text{St} = f(\dot{Q}_2/\dot{Q}_1, \text{Re}). \tag{6}$$

Hence

$$\text{Fr} \propto \text{St}^{1/3} \tag{7}$$

for a specific system in which the role of Reynolds number is insignificant.

It is also possible to conceive of the flow $\dot{V}$ as arising from the imposition of the temperature differential $\Delta T = T_2 - T_1$. Thus

$$\dot{V} \sim \Delta T, \beta gh, \dot{Q}_2/\dot{Q}_1, A, K \tag{8}$$

giving rise to the dimensionless result

$$\frac{(\dot{V}/A)^2}{\beta gh \Delta T} = f(\dot{Q}_2/\dot{Q}_1, K)$$

and

$$\frac{\text{Fr}^2}{\Delta T/T} = f(\dot{Q}_2/\dot{Q}_1, \text{Re}). \tag{9}$$

Then for a specified system in which Reynolds number does not play an important role

$$\text{Fr}^2 \propto \Delta T/T \tag{10}$$

and it follows from equation (7) that

$$\Delta T/T \propto \text{St}^{2/3}. \tag{11}$$

3. THE ROLE OF REYNOLDS NUMBER

The results (7), (10) and (11) provide the most primitive representation of the scale-dependence of this kind of system. Before seeing how they accord with reality, we shall consider modifications which express the effects of Reynolds number. These may be expected to be modest in comparison with the effects of Froude number, for the flow-resisting losses arising in this high-Reynolds-number flow may be expected to be rather weakly dependent upon Reynolds number. For turbulent flow in a smooth-walled channel, for example, $K \propto \text{Re}^{-1/5}$ indicates the nature of the dependence. In a rough-walled passage or one with sharp corners at which the flow is induced to separate, the dependence upon Re is even weaker.

Adopting the approach commonly followed in developing simple scaling laws in the absence of a detailed solution to a complex flow problem, we postulate that the effects of Reynolds number can be accounted for by a power-law factor introduced in equation (7). Thus

$$\text{Fr}\,\text{Re}^p \propto \text{St}^{1/3}. \tag{12}$$

The form of the corresponding generalisation of equation (10) can be obtained by observing that the basic energy relation

$$\dot{Q} \sim \rho c_p \dot{V} \Delta T$$

implies that

$$\text{St} \sim \text{Fr}\,\Delta T/T. \tag{13}$$

Combined with equation (12) this gives:

$$\Delta T/T \propto \text{Re}^p\,\text{St}^{2/3}. \tag{14}$$

Thus the single index p serves to define the role of Reynolds number, and it is to be anticipated that $p \ll 1$, consistent with the modest role expected for viscosity in this class of motions.

4. COMPARISONS WITH MEASUREMENT

A series of measurements carried out in a particular stairwell or stairwell model with various heat inputs $\dot{Q}$ serves as a means of assessing the efficacy of these results of dimensional analysis. Equations (7) and (11) imply that

$$\dot{V} \propto \dot{Q}^{1/3} \quad \text{and} \quad \Delta T/T \propto \dot{Q}^{2/3} \tag{15}$$

while the more general equations (12) and (14) imply that:

$$\dot{V} \propto \frac{1}{\dot{Q}^{3(1+p)}} \quad \text{and}$$

$$\Delta T/T \propto \dot{V}^p \dot{Q}^{2/3} \propto \dot{Q}^{(p+2/3)/(p+1)}. \quad (16)$$

The experiments carried out on a one-half scale model will be reported elsewhere. Here we need note only that they indicate that

$$\dot{V} \propto \dot{Q}^{0.25}, \quad \Delta T/T \propto \dot{Q}^{0.82}. \quad (17)$$

Greater confidence can be placed in the first of these results, since the determination of the temperature difference which characterises the process is not straightforward, involving as it does the determination of single temperatures typifying above-stairs and below-stairs conditions.

Evidently the primitive results (15) are not grossly incorrect. However, the results (16) are able to provide a superior representation of the experimental results. Comparison of the first of equations (16) and the first of equations (17) suggests that

$$p = 1/3. \quad (18)$$

The power appearing in the second of equations (16) is then 3/4, somewhat different from the value 0.82 suggested by the experiments, but a marked improvement on the value 2/3 given by the primitive analysis.

Adopting the value of p given in equation (18), we obtain the specific results

$$\mathrm{Fr}\,\mathrm{Re}^{1/3} \propto \mathrm{St}^{1/3} \quad (19)$$

$$\Delta T/T \propto \mathrm{Re}^{1/3}\,\mathrm{St}^{2/3}. \quad (20)$$

5. ALTERNATIVE FORMULATIONS

While the preceding formulae do serve to indicate the leading features of the system under consideration, they can be recast in forms that display the relationships more clearly. An appropriate Grashof number for the system is

$$\mathrm{Gr} = \frac{g\beta\Delta T A h}{\nu^2} = (\mathrm{Re}/\mathrm{Fr})^2 \Delta T/T = S^2 \Delta T/T \quad (21)$$

where

$$S = \mathrm{Re}/\mathrm{Fr} = A^{1/2}(gh)^{1/2}/\nu \quad \text{and} \quad \mathrm{Re} = \frac{\dot{V}}{\nu A^{1/2}} \quad (22)$$

is a characteristic Reynolds number. Note that the parameter $S = \mathrm{Re}/\mathrm{Fr}$ is a characteristic of the system and not of the 'operating point'.

Results (19) and (20) can now be rewritten as

$$\mathrm{Gr} \propto \mathrm{Re}^3 = S^3\,\mathrm{Fr}^3 \quad (23)$$

$$\mathrm{St} \propto \mathrm{Re}^4/S^3 = S\,\mathrm{Fr} \quad (24)$$

$$\mathrm{Gr} \propto (S^3\,\mathrm{St})^{3/4}. \quad (25)$$

From these expressions, we obtain directly the power-law relationships between the leading variables:

$$\Delta T \propto \dot{V}^3, \quad \dot{V} \propto \dot{Q}^{1/4}, \quad \Delta T \propto \dot{Q}^{3/4} \quad (26)$$

for a particular system. These formulae are, of course, very similar to the results of experiment [equation (17)] which provided guidance in their derivation.

6. THE EFFECTS OF SCALE

Two factors under the control of the experimenter are the size of the model and the magnitude of the energy input driving the motion. The first is, of course, decided once-for-all in the construction of each model, while the latter can readily be altered during a series of experiments. Indeed, an important function of the preceding dimensional considerations is their linking of the effects of size and energy input, so that variations in the conveniently changed parameter (energy input) can be used to infer the effects of geometric scale.

In passing, we may note that it is possible to construct a distorted model, in which vertical and horizontal length scales differ. However, an investigation based on the laws developed above reveals no advantage that would compensate for the evident loss of fidelity in the modelling of the geometry-dependent loss characteristics of the flow passages.

In discussing the effects of scale we shall consider three ratios of model to prototype properties:

$$l = \frac{\text{model dimensions}}{\text{prototype dimensions}} = h_m/h = (A_m/A)^{1/2} \quad (27)$$

$$q = \frac{\text{model energy input}}{\text{prototype energy input}} = \dot{Q}_m/\dot{Q} \quad (28)$$

$$f = \frac{\text{model Froude number}}{\text{prototype Froude number}} = \mathrm{Fr}_m/\mathrm{Fr}. \quad (29)$$

Taking the physical properties of the circulating fluid to be essentially constant and independent of the energy input and the response to it, we can express the ratios of all the quantities arising in this analysis in terms of the first two ratios:

$$\left.\begin{aligned}
S_m/S &= l^{3/2} &&= l^{3/2} \\
\mathrm{St}_m/\mathrm{St} &= q l^{-5/2} &&= f^4 l^{3/2} \\
\mathrm{Gr}_m/\mathrm{Gr} &= q^{3/4} l^{3/2} &&= f^3 l^{9/2} \\
\mathrm{Re}_m/\mathrm{Re} &= q^{1/4} l^{1/2} &&= f l^{3/2} \\
\mathrm{Fr}_m/\mathrm{Fr} &= q^{1/4} l^{-1} &&= f \\
\dot{V}_m/\dot{V} &= q^{1/4} l^{3/2} &&= f l^{5/2} \\
\dot{Q}_m/\dot{Q} &= q &&= f^4 l^4 \\
\Delta T_m/\Delta T &= q^{3/4} l^{-3/2} &&= f^3 l^{3/2} \\
u_m/u &= q^{1/4} l^{-1/2} &&= f l^{1/2}.
\end{aligned}\right\} \quad (30)$$

The third column in this set of results is obtained by noting that $f = q^{1/4}/l$ implies that $q = f^4 l^4$. These results are useful in indicating the effects on other parameters of deviations of the Froude number—the parameter dominating the dynamics of the flow—from its prototype value.

Tables 1 and 2 give evaluations of the ratios of equations (30) for two geometric ratios, one-half and one-tenth scale. These happen to be the scales that have been adopted in the experimental programme from which this work springs, but they are of more general interest in

Table 1. Changes in parameters for one-half-scale model ($l = \frac{1}{2}$, $S_m/S = 0.354$)

$Fr_m/Fr = f$	1	1.3	1.4	1.6
Re_m/Re	0.354	0.460	0.495	0.566
Gr_m/Gr	0.0442	0.0971	0.1213	0.1817
St_m/St	0.354	1.010	1.358	2.320
$\dot{V}_m/\dot{V}$	0.1768	0.2298	0.2474	0.2828
$\dot{Q}_m/\dot{Q} = q$	0.0625	0.1785	0.2401	0.4096
$\Delta T_m/\Delta T$	0.354	0.777	0.970	1.448
u_m/u	0.707	0.919	0.990	1.131

Table 2. Changes in parameters for one-tenth-scale model ($l = 1/10$, $S_m/S = 0.0316$)

$Fr_m/Fr = f$	1	3	4	6
Re_m/Re	0.0316	0.0949	0.1265	0.1896
Gr_m/Gr	0.0000316	0.000854	0.00202	0.00682
St_m/St	0.0316	2.187	12.83	40.95
$\dot{V}_m/\dot{V}$	0.00316	0.00949	0.01265	0.01897
$\dot{Q}_m/\dot{Q} = q$	0.00010	0.0081	0.0256	0.1296
$\Delta T_m/\Delta T$	0.0316	0.854	2.024	6.826
u_m/u	0.316	0.949	1.265	1.897

spanning the range of scales likely to be encountered in practice.

An inspection of Table 1 reveals no substantial difficulties attendant upon the adoption of half-scale modelling. The choice of that particular energy input which generates a Froude number identical to that of the prototype (that is, $f = 1$) gives rise to a Reynolds number about one-third of the prototype value; such a ratio is likely to be acceptable for a high-Reynolds-number flow in which the losses are only weakly dependent upon viscosity. A more serious matter is the fall of the model temperature variations in the same ratio as the Reynolds number, to about one-third the prototype values. Since the prototype temperature variations are only a few degrees, this implies that the model temperature variations will be of order one degree. Since the temperatures, like other properties of the flow, display wide fluctuations in time, this points to considerable difficulty in measurement of the temperature field. For the choice $f = 1$ the velocities of the model are some one-third below the prototype values; this too indicates difficulties in measurement, the velocities to be measured being of order $\frac{1}{2}$ m/s or less.

The difficulties of measurement identified in the preceding paragraph are alleviated (not eliminated, since even the prototype values of temperature and velocity cannot be measured easily) by increasing the heat input to about four times the value necessary for $f = 1$, to give a Froude number ratio of $f = 1.4$. When this is done, the magnitudes of both temperature and velocity fields are identical for model and prototype. A small move of the Reynolds number towards the prototype value is also achieved, the model value being about one-half that for the prototype. Since the experimeter will not wish to operate his model system with values of Froude and Reynolds numbers too different from those of the prototype, it can be argued that the choice $q = \frac{1}{4}$ for $l = \frac{1}{2}$ provides a nice balance between the necessarily conflicting aims of experimental design.

Consideration of Table 2 suggests that the choice $f = 1$ corresponds to an energy input of only a fraction of one watt, giving rise to temperature variations of small fractions of one degree. The velocities too are very small, and the Reynolds number is only a few percent of the prototype value. Evidently these conditions are not conducive to the extraction of meaningful experimental results. Rather more useful experiments appear to be possible for $f = 3$ or 4. Here the Reynolds number is around one-tenth the prototype value, the velocities and temperature variations are as large as or greater than the prototype values, and the energy input will be of order 100 W, sufficient to dominate the effects of varying environmental conditions. A further increase in energy input, to give $f = 6$, gives even more easily measured temperatures and velocities but at the price of moving still further from the prototype 'operating point'. In extending the scaling laws that far, we should perhaps be investing in them greater confidence than they deserve. It must be admitted that it is not easy to identify a satisfactory operating régime for a model this small. Perhaps its greatest value is in flow visualisation and preliminary tests upon whose quantitative results too much confidence need not be placed.

It is tempting to consider the feasibility of using models somewhat smaller than half scale, but sufficiently larger than one-tenth scale to avoid the worst of the problems identified there in Table 2. Note that the quantities of material and floor areas required for one-third and one-quarter scale models are, respectively, around one-half and one-quarter those for a half-scale model. The formulae (30) provide the means of weighing up the relative merits and demerits of such intermediate models.

7. CONCLUSIONS

1. For the simplest case of stairwell flow, namely, the case of quasi-steady fully recirculating flow, the gross features of the flow are described by relationships between a Grashof number Gr (or alternatively the temperature ratio $\Delta T/T$), a Froude number Fr (expressing the balance between buoyancy and inertia forces), a Reynolds number Re (introducing the influence of viscosity on the resistance to the flow), and the ratio of energy extractions from the system above and below the stairwell, $\dot{Q}_2/\dot{Q}_1$.

2. If the role of viscosity is assumed negligible, the form of the relationships for a specified value of the ratio $\dot{Q}_2/\dot{Q}_1$ can be determined as simple power laws, using dimensional arguments alone.

3. Experiments on a single experimental configuration, but with varying energy input, serve to determine the Reynolds-number effect, which can be expressed through an additional power-law factor, to the accuracy consistent with the available data.

4. The relationships thus generated are vital adjuncts of experimental design, providing the means of assessing the effects of changes in the model size and in the energy input to the model.

5. For one-half-scale models it is possible to determine operating conditions which preserve the essential dynamical features of the motion, while generating velocity and temperature fields that can readily be inves-

tigated using available measuring equipment. Models of this scale, or perhaps a little smaller, can thus be used to determine the quantitative behaviour of prototype stairwell flows.

6. For one-tenth-scale models it is not obvious that there exist operating conditions which retain the essential dynamic features of the prototype flows, while at the same time generating velocity and temperature fields that can be measured with confidence. The role of models of such small scales is thus likely to be in general investigations, possibly using flow visualisation.

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