Building and Environment, Vol. 25, No. 4, pp. 339–347, 1990.
Printed in Great Britain.

0360–1323/90 $3.00+0.00

\# 4538

A Multi-Chamber Ventilation Model with Random Parameters

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A generalized multi-chamber ventilation model is developed for air contaminant prediction problems where the parameters of the system, such as airflow rates, are described by Gaussian probability distributions. A numerical solution, utilizing stochastic differential equations (SDE's), is provided to facilitate its application. The model is used to calculate contaminant concentration histories described by means and standard deviations. It is also used to show the sensitivity of concentrations to the variation of such parameters as infiltration flows and contaminant source rates. Sample applications of the model are provided.

NOMENCLATURE

B	covariance parameter matrix
COV(.)	covariance operator
C_i	concentration of contaminant in cell i
E	expectation operator
F	drift or mean coefficient matrix
G	diffusion coefficient matrix
K_i	time invariant coefficient i
ODE	ordinary differential equation
Q_{C_i}	contaminant production in cell i
Q_{ij}	airflow from cell i to cell j
SDE	stochastic differential equation
T	matrix transpose operator
tr	matrix trace operator
V_i	volume of cell i
VAR(.)	variance operator
δ	Dirac delta function
ξ	matrix of Gaussian white noise random variables
Φ	Itô formula variable

INTRODUCTION

IN THE analysis and design of ventilation systems, there are inherent variabilities in such parameters as air contaminant production and ventilation airflows which need to be included in any realistic model of air contaminant concentrations. For example, consider a ventilated room where smoking is permitted. The production of cigarette smoke will be highly variable, due to dependence on the number of smokers in the room. Additionally, the amount of fresh air supplied to the room will be uncertain. This will depend on such factors as doors and windows being opened or closed, and the operation of the mechanical ventilation system. A further source of uncertainty is variable weather patterns which affect infiltration/exfiltration flows.

A method of dealing with these uncertainties in a ventilation system model is to use stochastic differential equations (SDE's) [1–3]. SDE's are the stochastic analogue of ordinary differential equations (ODE's), allowing initial conditions and coefficients representing physical parameters in the model to be described by Gaussian probability distributions.

The application of SDE theory to ventilation problems is a recent occurrence, and there are a number of concerns related to the application which need to be resolved. Except for very simple problems which have analytical solutions, the use of SDE's is inhibited by the necessity of solving a complicated set of coupled moment equations. Numerical formulation and solution of the equations would facilitate the application of SDE models. Further research areas relate to the appropriate modeling of the random variation in ventilation parameters, and the accuracy of SDE model solutions compared to actual air contaminant concentration histories.

This paper provides a numerical solution to a generalized SDE model of ventilation, formulated within the framework of a multi-chamber model, which treats air spaces as ideally mixed cells. The solution is a Gaussian stochastic process, with a probability distribution described by its first and second order moments.

The applications of this model are expected to be useful in several areas. The method allows statements to be made about the probability of contaminant concentration levels exceeding a regulatory level. The effect on contaminant concentration due both to changes in mean values and uncertainty levels of ventilation parameters can be investigated in sensitivity studies. Furthermore, the convenient numerical solution will facilitate future experimental investigations of the model.

A MULTI-CHAMBER SDE MODEL

When modeling the transport of a passive air contaminant in a building, it is common to treat the building as a set of inter-connected chambers or mixing cells [4]. The instantaneously and uniformly mixed cells are generally taken to represent rooms in a building.

Uniform mixing within a cell is an idealization which has been justified for many practical problems. For ex-

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ample, tracer gas experiments conducted in a research house showed that individual rectangular rooms with normal ceiling heights (about 2.5 m) could be treated as uniformly mixed [5]. Between-room contaminant concentration differences were shown to be more significant than within-room spatial variations. The multi-chamber model was also shown to be useful in experiments using a five room test house [6].

The occurrence of short-circuiting, the presence of stagnant zones, and the existence of multiple airflow regions are possible causes if incomplete mixing does occur [7]. When incomplete mixing is significant, mixing coefficients can be introduced to replace the complete mixing assumption within a cell while maintaining the simplicity of the mixing cell approach [8]. Division of a room into two or more mixing cells is an effective approach when required [9].

The deterministic model

A generalized deterministic multi-chamber model is represented here by a species conservation equation set consisting of N first order, linear, coupled, ODE's:

$$\begin{array}{llllll} K_1C_1 & + K_2C_2 & + \dots & K_NC_N & + K_{N+1} \\ K_{N+2}C_1 & + K_{N+3}C_2 & + \dots & K_{2N+1}C_N & + K_{2N+2} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ = K_{N^2}C_1 & + K_{N^2+1}C_2 & + \dots & K_{N^2+N-1}C_N & + K_{N^2+N} \end{array} \quad (1)$$

his model, there are N cells which could, for ex-, represent rooms in a building which are exchang-irflows between themselves and the environment. concentration of a passive air contaminant in each e N cells is identified by the C_i terms. The species be transferred from one cell to any other cell. This t is accounted for by the K_i constant coefficients tiplying the concentrations on the right hand side of ations (1). These coefficients typically represent inter-l airflows and may also include mixing factors. The t K_i term in each equation represents parameters which e not involved in the transport of the species between ells, such as the production of the species in a cell or the emoval of the species to the environment.

The stochastic model

In practice, the K_i coefficients representing physical parameters in equations (1) have random variations. Also, the initial species concentrations in the cells may be uncertain. If the random variations in the coefficients and initial conditions are modeled by Gaussian white noise, a set of stochastic differential equations (SDE's) results, the solution of which enables a description to be made of the probability distribution of the species concentration in each cell as a function of time. Like Gaussian white noise, the variations in the coefficients and initial conditions should approach the property of being uncorrelated with time. This is an idealization which never occurs in real ventilation systems. However, if the time scale of the noise autocorrelation function is much smaller than that of the solution process being calculated, it is a good approximation.

In the SDE model developed here, the parameters in the ventilation system, such as contaminant production rates, airflows, and initial conditions, are modeled by time-invariant means and standard deviations. As an illustration, consider an underground parking garage where carbon monoxide is the air contaminant. The production of carbon monoxide could be modeled by a constant mean and standard deviation on an hourly basis, but not over a daily time period when variations in the mean (due to weekends *etc*) would be expected. Furthermore, the Gaussian white noise approximation of the variability in carbon monoxide production would be appropriate as long as the time scale of the variability is in the order of seconds or minutes, as would be expected if the effect is due to individual cars.

To obtain the relevant SDE's, the coefficients in equations (1) are decomposed into mean and varying components as:

$$K_i = \bar{K}_i + K_i'. \quad (2)$$

The variable components will be modeled by a matrix of Gaussian white noise random variables (ξ), which has the following mean and correlation properties:

$$E(\xi) = 0,$$
$$E(\xi_t \xi_{t-s}) = \mathbf{B}\delta(t-s),$$

where E denotes expectation, t and s are time locations, δ is the Dirac delta function, and $\mathbf{B}$ is a covariance parameter matrix with entries which correspond to covariances of the K_i constants.

$\mathbf{B}$ is written as:

$$\begin{bmatrix} \Upsilon_{1,1} & \Upsilon_{1,2} & \Upsilon_{1,3} & \dots & \Upsilon_{1,N^2+N} \\ \Upsilon_{2,1} & \Upsilon_{2,2} & \Upsilon_{2,3} & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \Upsilon_{N^2+N,1} & \Upsilon_{N^2+N,2} & \Upsilon_{N^2+N,3} & \dots & \Upsilon_{N^2+N,N^2+N} \end{bmatrix}, \quad (3)$$

where $\Upsilon_{i,j} = \mathrm{COV}(K_i'K_j') = \mathrm{COV}(K_iK_j)$.

The K_i constants can be composed of a sum of any number of physical processes, however if there is a product or quotient of two or more physical processes which have random variations, they must be linearized in order to make the SDE tractable. For example: given

$$K = A * B,$$

where

$$A = \bar{A}+A' \quad \text{and} \quad B = \bar{B}+B',$$

then

$$K = \overline{AB}+\bar{A}B'+\bar{B}A'+A'B'$$
$$\approx \overline{AB}+\bar{A}B'+\bar{B}A'.$$

In this case, K is linearized by neglecting the $A'B'$ term. This is reasonable since it is generally much smaller than the other terms involving the mean values.

The covariance of the K_i terms may then be calculated using the rules for linear combinations of independent variates. Using the example above:

$$\begin{aligned}\mathrm{COV}(K,K) &= \mathrm{VAR}(K,K)\\ &= \bar{A}^2\mathrm{VAR}(B')+\bar{B}^2\mathrm{VAR}(A')\\ &= \bar{A}^2\mathrm{VAR}(B)+\bar{B}^2\mathrm{VAR}(A).\end{aligned}$$

When equation (2) is substituted into the deterministic equations (1), as SDE results, which can be written in matrix form as:

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t} = \mathbf{F}(\mathbf{C})+\mathbf{G}(\mathbf{C})\xi, \tag{4}$$

where:

$$\mathbf{C}^T = [C_1 C_2 \ldots C_N], \tag{5}$$

$$\xi^T = [K'_1 K'_2 \ldots K'_{N^2+N}], \tag{6}$$

$$\mathbf{F} = \begin{bmatrix} \bar{K}_1 C_1 & + & \bar{K}_2 C_2 & + \ldots & \bar{K}_N C_N & + \bar{K}_{N+1} \\ \bar{K}_{N+2} C_1 & + & \bar{K}_{N+3} C_2 & + \ldots & \bar{K}_{2N} C_N & + \bar{K}_{2N+2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \bar{K}_{N^2} C_1 & + & \bar{K}_{N^2+1} C_2 & + \ldots & \bar{K}_{N^2+N-1} & + \bar{K}_{N^2+N} \end{bmatrix}, \tag{7}$$

$$\mathbf{G} = \begin{bmatrix} C_1 \ldots C_N\, 1\; 0 \ldots 0\; 0\; 0 \ldots 0\; 0\; 0\; 0 \\ 0 \ldots 0\; 0\, C_1 \ldots C_N\, 1\; 0 \ldots 0\; 0\; 0\; 0 \\ \cdot \qquad\qquad\qquad\qquad \cdot \\ \cdot \qquad\qquad\qquad\qquad \cdot \\ \cdot \qquad\qquad\qquad\qquad \cdot \\ 0 \ldots 0\; 0\; 0 \ldots 0\; 0\; 0 \ldots C_1 \ldots C_N\, 1 \end{bmatrix}. \tag{8}$$

$\mathbf{C}$ is the concentration matrix, ξ is the Gaussian white noise matrix, $\mathbf{F}$ is the drift matrix, $\mathbf{G}$ is the diffusion matrix, and T represents the matrix transpose operation.

SOLVING FOR THE MOMENTS OF CONCENTRATION

A solution of the multi-chamber SDE ventilation model, represented by equation (4), is dependent upon an external interpretation of the resulting integral involving the Gaussian white noise term [10]. There are two interpretations, attributed to Itô and Stratonovich, which have gained popular acceptance in the application of SDE's to engineering problems [11–13].

Solving the SDE (4) using either the Itô or Stratonovich interpretations will result in a set of possible solutions. In other words, for a given probability w, a specific solution $\mathbf{C}(t)$ results. Changing the probability results in another solution. For the sake of modeling the possible outcomes of a physical system, it is usually of more interest to know the range of possible solutions. SDE solutions of contaminant concentration processes in the multi-chamber ventilation model developed here will consist of solving for the first two moments, and hence the mean and standard deviations of concentration as a function of time. The higher moments about the mean concentration will be equal to zero for the linear SDE's used in this model.

The main instrument used to solve for the moments of an SDE is Itô's formula [10]. The method of solution using the Itô formula is summarized here and is expressed in a generalized, multi-dimensional form. It states that given an SDE (4) (interpreted here as a Stratonovich equation since the Itô solution is easily related to the Stratonovich solution), there is a real-valued function of the solution process defined by $\Phi = \Phi(t, \mathbf{C}(t))$ which is obtained by solving the following differential:

$$\mathrm{d}\Phi = \Phi_t\,\mathrm{d}t+\Phi_c^T\left(\mathbf{F}+\frac{1}{2}\mathbf{GB}\frac{\partial \mathbf{G}}{\partial \mathbf{C}}\right)\mathrm{d}t + \frac{1}{2}tr\mathbf{GBG}^T\Phi_{cc}\,\mathrm{d}t+\Phi_c^T\mathbf{G}\xi\,\mathrm{d}t, \tag{9}$$

where T and tr denote matrix transpose and trace operations, and:

$$\Phi_t = \frac{\partial \Phi}{\partial t}, \tag{10}$$

$$\Phi_c^T = \left[\frac{\partial \Phi}{\partial C_1}\frac{\partial \Phi}{\partial C_2}\cdots\frac{\partial \Phi}{\partial C_N}\right], \tag{11}$$

$$\Phi_{cc} = \begin{bmatrix} \frac{\partial^2\Phi}{\partial C_1^2} & \frac{\partial^2\Phi}{\partial C_1 \partial C_2} & \cdots & \frac{\partial^2\Phi}{\partial C_1 \partial C_N} \\ \frac{\partial^2\Phi}{\partial C_2 \partial C_1} & \frac{\partial^2\Phi}{\partial C_2^2} & \cdots & \frac{\partial^2\Phi}{\partial C_2 \partial C_N} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial^2\Phi}{\partial C_N \partial C_1} & \frac{\partial^2\Phi}{\partial C_N \partial C_2} & \cdots & \frac{\partial^2\Phi}{\partial C_N^2} \end{bmatrix}. \tag{12}$$

At this point the functional relation of Φ can be chosen. To simplify the solution of the first and second moments we can set:

$$\Phi = C_1^a C_2^b, \ldots, C_N^z, \qquad a, b, \ldots, z \in (0,1,2).$$

This simplifies the Itô formula since Φ_t, equation (10), equals zero for this case. The moment equations are derived by setting all possible combinations of the

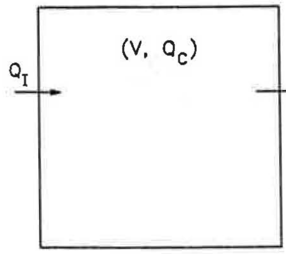

Fig. 1. 1-Cell model.

exponents $a, b, \ldots, z$ such that $(a+b+,\ldots,z)$ = the desired moment.

To solve for the moments, the expectation of the Itô formula, equation (9), is taken:

$$E\left[\frac{d(C_1^a C_2^b,\ldots,C_N^z)}{dt}\right] = E[\boldsymbol{\Phi}_c^T \mathbf{F}] + E\left[\boldsymbol{\Phi}_c^T \frac{1}{2} \mathbf{GB} \frac{\partial \mathbf{G}}{\partial \mathbf{C}}\right] + E\left[\frac{1}{2} tr \mathbf{GBG}^T \boldsymbol{\Phi}_{cc}\right] + E[\boldsymbol{\Phi}_c^T \mathbf{G}\xi]. \quad (13)$$

The last term on the RHS of equation (13) is equal to ıce the expectation of the Gaussian white noise ; (ξ) is equal to zero and the three terms in the ,sion are statistically independent. Finally, the ssion for the expectation of the Itô formula, written mmation notation becomes:

$$\left.\frac{d(C_1^a C_2^b,\ldots,C_N^z)}{dt}\right]$$

$$= E\left[\sum_{i=1}^{N} \frac{\partial C_1^a C_2^b,\ldots,C_N^z}{\partial C_i}\left(f_i + \frac{(\mathbf{GBG}_c)_i}{2}\right)\right] + E\left[\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 C_1^a C_2^b,\ldots,C_N^z}{\partial C_i \partial C_j} (\mathbf{GBG}^T)_{i,j}\right], \quad (14)$$

where f_i represents the rows of the drift matrix $\mathbf{F}$ as defined in equation (7) and:

$$(\mathbf{GBG}_c)_i = \sum_{k=1}^{N}\sum_{l=1}^{N^2+N}\sum_{m=1}^{N^2+N} \mathbf{G}_{kl}\mathbf{B}_{lm}\frac{\partial \mathbf{G}_{im}}{\partial \mathbf{C}_k}, \quad (15)$$

$$(\mathbf{GBG}^T)_{i,j} = \sum_{k=1}^{N^2+N}\left(\sum_{l=1}^{N^2+N} \mathbf{G}_{il}\mathbf{B}_{lk}\right)\mathbf{G}_{kj}^T. \quad (16)$$

If the Itô interpretation were taken, the $(\mathbf{GBG}_c)_i$ terms in equation (14) would not be present.

Itô vs Stratonovich solutions

A discussion of the advantages and disadvantages of using either the Itô or Stratonovich interpretations as models of physical systems can be found in [14]. Here it is shown that for most practical ventilation problems, the difference between the Itô and Stratonovich solutions is insignificant. A one-cell problem illustrates this.

A single ventilated chamber shown in Fig. 1 has an inlet flow (Q_I m^3 h^{-1}), an exhaust flow (Q_E m^3 h^{-1}), a contaminant production term (Q_C m^3 h^{-1}), a volume (V m^3), and an inlet flow contaminant concentration (C_I m^3 m^{-3}(air)). The deterministic contaminant mass balance is expressed as:

$$\frac{dC}{dt} = -\frac{Q_E}{V}C + \frac{(C_I Q_I + Q_C)}{V} = K_1 C + K_2. \quad (17)$$

Here the volume of air in the cell is considered to be equal to the volume of the cell. This causes errors of less than 1% for air contaminant concentrations up to 10 000 ppm [15].

Uncertainty in Q_I, Q_E, and Q_C is introduced by modeling them as having mean and random components as:

$$Q_i = \bar{Q}_i + Q_i'. \quad (18)$$

While Q_I', Q_E, and Q_E' will in practice be correlated to some degree through the mass balance, the present model does not take this into account.

When these substitutions are made and the random components are modeled as Gaussian white noise, an SDE results:

$$\frac{dC}{dt} = -\frac{\bar{Q}_E}{V}C + \frac{(C_I\bar{Q}_I + \bar{Q}_C)}{V} - \frac{Q_E'}{V}C + \frac{(C_I Q_I' + Q_C')}{V} = \bar{K}_1 C + \bar{K}_2 + K_1' C + K_2'. \quad (19)$$

The first and second order moments of the SDE will be solved for both the Stratonovich and Itô interpretations in order to compare the two. After specifying the inlet flow concentration of contaminant and initial conditions to be zero, the first moment Stratonovich solution is obtained as:

$$\bar{C} = \frac{K_2}{K_1 + \frac{\mathrm{VAR}(K_1)}{2}}\left[\exp\left[K_1 + \frac{\mathrm{VAR}(K_1)}{2}\right]t - 1\right], \quad (20)$$

where:

$$\mathrm{VAR}(K_1) = \frac{\mathrm{VAR}(Q_E)}{V^2}. \quad (21)$$

Note that VAR(Q_E) is the variance parameter of a Gaussian white noise process representing the random variations of the exhaust flow rate. It has the same numerical value as the variance of the exhaust flow rate, but has units of m^6 h^{-1}.

The difference between the Itô and Stratonovich first moment solutions for this one cell problem is due to the two VAR(K_1)/2 terms in equation (20). If these terms are neglected, the Itô mean solution is obtained, which is identical to the deterministic solution of equation (17).

The second moment Stratonovich solution is:

$$\overline{C^2} = \frac{2\bar{K}_2^2}{\left(\bar{K}_1 + \frac{\mathrm{VAR}(K_1)}{2}\right)\left(-\bar{K}_1 - \frac{3}{2}\mathrm{VAR}(K_1)\right)} \times \left[\exp\left[\bar{K}_1 + \frac{\mathrm{VAR}(K_1)}{2}\right]t - \exp\left[2\bar{K}_1 + 2\mathrm{VAR}(K_1)\right]t\right] + \frac{\mathrm{VAR}(K_2) - \frac{2\bar{K}_2^2}{\left(\bar{K}_1 + \frac{\mathrm{VAR}(K_1)}{2}\right)}}{2\bar{K}_1 + 2\mathrm{VAR}(K_1)} \times [\exp[2\bar{K}_1 + 2\mathrm{VAR}(K_1)]t - 1], \quad (22)$$

where:

$$\mathrm{VAR}(K_2) = \frac{C_1^2\mathrm{VAR}(Q_1)+\mathrm{VAR}(Q_C)}{V^2}. \quad (23)$$

The second moment Itô solution is:

$$\overline{C^2} = \frac{2\bar{K}_2^2}{(\bar{K}_1)(-\bar{K}_1-\mathrm{VAR}(K_1)}$$
$$\times [\exp[\bar{K}_1]t - \exp[2\bar{K}_2+\mathrm{VAR}(K_1)]t]$$
$$+ \frac{\mathrm{VAR}(K_2)-\dfrac{2\bar{K}_2^2}{K_1}}{2\bar{K}_1+\mathrm{VAR}(K_1)}[\exp[2\bar{K}_1+\mathrm{VAR}(K_1)]t-1]. \quad (24)$$

The differences between the Itô and Stratonovich second moment solutions are again due to additional $\mathrm{VAR}(K_1)$ terms. To investigate the effect of these terms on the Itô and Stratonovich solutions for a typical ventilation problem, the single cell is considered to represent a room having four air changes per hour ($Q_{I/E}/V = 4$ h^{-1}). A source is producing contaminant (Q_C) at a rate equal to 2% of the inlet or exhaust flows ($Q_{I/E}$). The standard deviations of Q_I, Q_C, and Q_E are assumed to be equal to one tenth of their mean values on an hourly basis. While Q_I must equal Q_E in a 1-cell-model, it is not necessary that $\mathrm{VAR}(Q_I)$ equal $\mathrm{VAR}(Q_E)$ in a particular case. Under these circumstances, the Stratonovich mean and second moment solutions are only 2% greater than the corresponding Itô solutions, a negligible difference. Numerical solutions of multi-cell ventilation problems developed later, also indicate that the Itô and Stratonovich solutions are not significantly different.

NUMERICAL SOLUTION OF THE MOMENTS

In general, the N-cell multi-chamber SDE model developed here will require the solution of $((N^2+N)/2+N)$ coupled ODE's in order to find the first and second moments of concentration in each cell. The complexity of these equations deters the use of SDE's for all but very simple problems, which often have analytical solutions. A numerical procedure to formulate and solve the governing first and second order moment equations would make the application of the model more attractive.

Through a study of patterns in the coefficients generated in the two moment equations, it is possible to generate first and second order moment equations for a general N-cell problem [16]. These are solved simultaneously using a 4th order Runge–Kutta routine in two FORTRAN subroutines. One subroutine, STRATSOLVE, solves using the Stratonovich interpretation of the SDE while the other subroutine, ITOSOLVE, solves using the Itô interpretation. Both subroutines calculate the mean and standard deviation of concentration in each cell at specified time locations.

It is noted that although the multi-chamber SDE model was developed specifically with a ventilation system in mind, the STRATSOLVE and ITOSOLVE subroutines can be used for other engineering problems [11, 12]. In fact, any problem which can be expressed in the form of equation (6) (in a deterministic sense) is compatible with the programmed solution developed here. If the K_i constants and their covariances can be specified, along with initial conditions, then a solution for the first and second order moments can be generated.

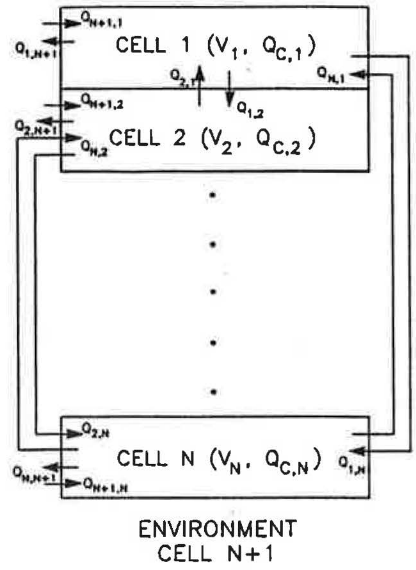

Fig. 2. N-cell ventilation model.

Two mainline programs, STRAVEN and ITOVEN, which utilize the STRATSOLVE and ITOSOLVE subroutines respectively, were created to solve for the Stratonovich and Itô solutions of a specific N-cell ventilation model specified in Fig. 2. This model allows for airflows, Q_{ij}, between all cells (including the environment). As well, each cell has a specified volume, V, and contaminant production rate, Q_{C_i}. The parameters which can have uncertainty in this model include the airflows, contaminant production rates, and initial conditions. All that is required is the minimum problem specification data. Further detail about these programs is available in [16]. Also, copies may be obtained from the authors.

APPLICATIONS

To illustrate the application of the multi-chamber SDE model, a 2-cell problem and a 6-cell problem are considered. For comparison, solutions of the following problems were found for both the Stratonovich and Itô interpretations using the STRAVEN and ITOVEN programs. It was found that the two solutions varied by less than 1.5% for both examples.

2-Cell problem

Figure 3 illustrates the basic 2-cell model and Table 1 provides the relevant input data. For this example, two equal sized rooms (100 m^3) are represented by the two cells. Air of zero contaminant concentration is supplied to room 1 at a rate of 200 m^3 h^{-1}, and air is exhausted from room 2 at the same rate, creating a net flow of air from room 1 to room 2. It is assumed that the two rooms are connected by a doorway and that secondary airflows cause significant mixing between the rooms. This is represented by equal and opposite flows of 400 m^3 h^{-1} between the rooms. The only uncertainty in this problem is in the initial concentration of a contaminant, such as cigarette smoke, in room 1. This is represented by a mean and

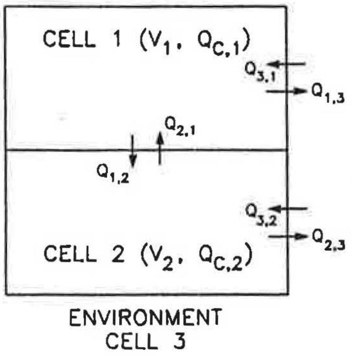

Fig. 3. 2-Cell mixing model.

Table 1. Data for 2-cell problem

Parameter	Room 1 (mean)	Room 1 (SD)	Room 2 (mean)	Room 2 (SD)
V (m^3)	100	—	100	–
C_{initial} (ppm)	1000	100	0	0
$Q_{C,i}$ (m^3 h^{-1})	0	0	0	0
$Q_{i,1}$ (m^3 h^{-1})	—	—	400	0
m^3 h^{-1})	600	0	—	—
h^{-1})	0	0	200	0
h^{-1})	200	0	0	0

ɔly air concentration C_3 (ppm) = 0.

lard deviation of 1000 and 100 ppm respectively. ɪm 2 is initially a clean room.

: is of interest to examine the effect of the initial ɔertainty in the contaminant concentration of room 1. ɪis is illustrated in Figs 4 and 5, which show the mean ɪd 95% confidence limits of concentration (mean ± two tandard deviations) for the two rooms. The con- ɔentration in room 1 decreases exponentially to zero as contaminant is transferred to room 2, where it is exhausted. The uncertainty level of contaminant concentration in room 1, defined here as the ratio of the standard deviation to the mean, retains its initial level of 10% throughout the process. This is expected since there is no uncertainty in any of the other model parameters. Room 2 responds as though it were exposed to an initial pulse of contaminant. The concentration increases to a maximum at about 15 min, and then decreases to zero. The ratio of the standard deviation to the mean in room 2 also remains constant at 10%, because room 1 is the only source of contaminant.

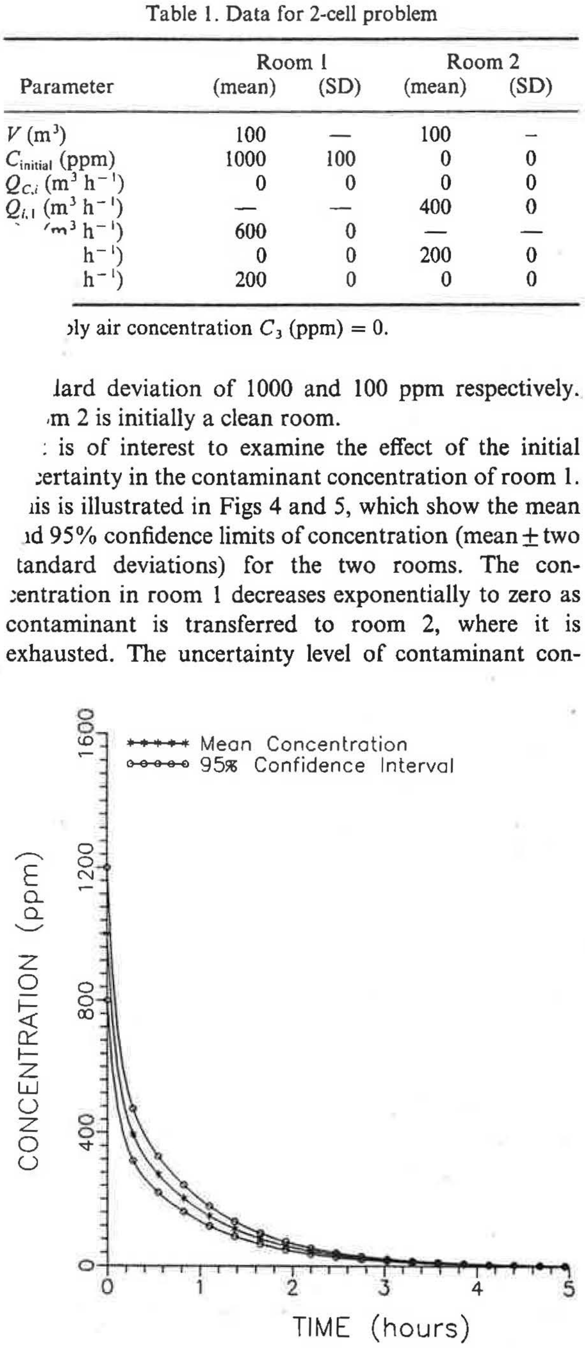

Fig. 4. Room 1 concentration history.

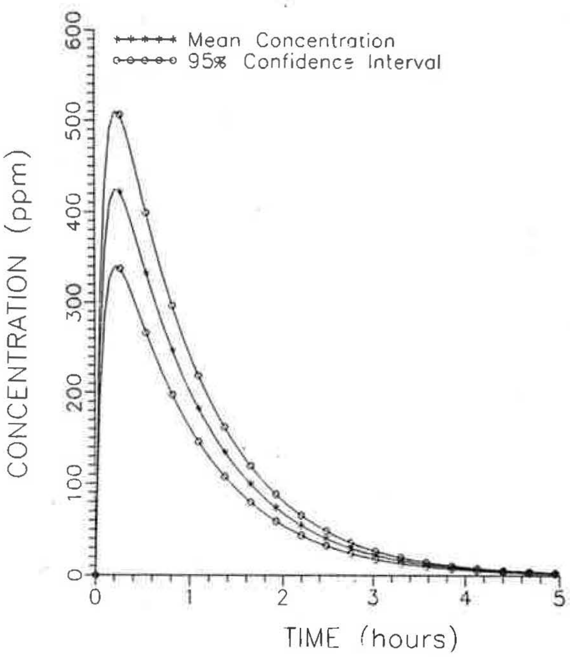

Fig. 5. Room 2 concentration history.

6-Cell problem

Figure 6 shows the floor plan of a single storey community hall building which is used for general meetings and receptions. The building is dilution ventilated in order to control the concentration of carbon dioxide, the contaminant to be modeled. There are in total, six interior rooms or cells, including the main hall area, kitchen, entrance, two restrooms, and lounge (numbered one through six cells). The environment can be thought of as a seventh cell which has a constant concentration of carbon dioxide.

The physical data relevant to the problem is presented in Table 2. The number written in brackets after the name of each room indicates the average number of people expected to occupy the room for a typical gathering. These numbers are used as the basis for calculating the production of carbon dioxide in each room and the required supply of outdoor air, following standard guidelines [17]. These guidelines give relevant data to the problem, including carbon dioxide generation rate per persion (0.0178 m^3 h^{-1}), outdoor air requirements for each type of room (non-smoking assumed), and the environmental carbon dioxide concentration (300 ppm).

The design problem is to determine the expected levels of carbon dioxide concentrations in the building. A probability distribution for the concentration of carbon dioxide in each room is desirable in order to make some

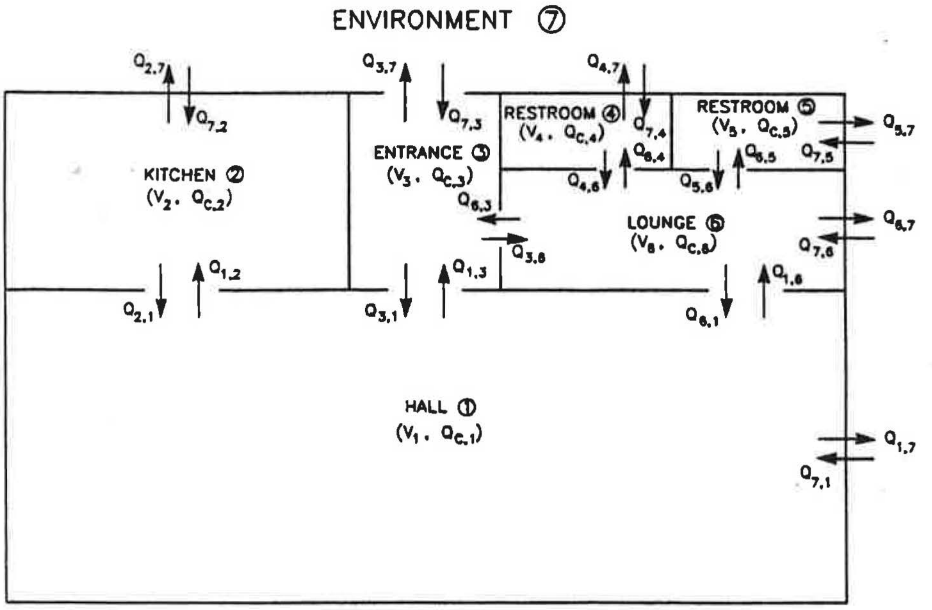

Fig. 6. A six room community hall building.

reasonable statements about the possibility of exceeding safe exposure levels. This can be accomplished with the multi-chamber SDE model given the mean and standard deviations of all flow rates, contaminant production rates, and initial conditions.

There are two main sources of uncertainty in this problem. Neither the number of people attending a gathering nor the number of people present in any room is constant. Thus, the production of carbon dioxide is variable. Also, there is a high degree of uncertainty in airflows through doorways since doors may be opened or closed, and the flows are affected by temperature and pressure gradients, and human activity.

It is possible to determine air exchange through doorways based on ventilation flows and temperature and pressure gradients [6]. For the purpose of this problem it will be assumed that there are no temperature or pressure gradients driving inter-room flows, and the flows through the doorways will be estimated using typical turbulent room air velocities [18]. Using a mean turbulent air velocity of 0.15 m s^{-1} to model the diffusion in both directions, door areas of 2 and 4 m^2, and an assumption that the doors are open 10% of the time, inter-cell airflows are calculated as 108 and 216 m^3 h^{-1} for single and double doors respectively. The standard deviations of the contaminant production rates and doorway flows (except those from the lounge to the restrooms) are set at 10% of their mean values for illustration purposes.

The mean and standard deviations of the concentration of carbon dioxide in each room as functions of time are shown in Figs 7 and 8. Using this information, statements can be made about the probability of exposure to a certain concentration of carbon dioxide in any room. For example, the 95% confidence interval for the hall is plotted in Fig. 9. This indicates that an exposure to a concentration level greater than 1725 ppm has a 2.5% chance of occurring for this model.

Sensitivity analysis

The multi-chamber SDE model can be used to investigate how changes in the mean or uncertainty levels of ventilation parameters affect contaminant concentration levels in the cells. The 6-cell hall example is used here to illustrate a sensitivity application. The effect on the carbon dioxide concentration uncertainty level in the lounge due to changes in the uncertainty level of the contaminant production rate in the hall is considered. In case A, the standard deviation of the production of carbon dioxide in the hall is one tenth of the mean value as seen in Table 2. In case B it is changed to one third of the mean value. All other mean and standard deviations are as defined previously in Table 2.

Table 2. Data for 6-cell problem

Parameter	Hall (75) (mean)	(SD)	Kitchen (9) (mean)	(SD)	Entrance (6) (mean)	(SD)	Restrooms (2) (mean)	(SD)	Lounge (15) (mean)	(SD)
V (m^3)	450	—	108	—	54	—	18	—	72	—
$C_{initial}$ (ppm)	0	0	0	0	0	0	0	0	0	0
$Q_{C,i}$ (m^3 h^{-1})	1.340	0.134	0.161	0.0161	0.107	0.0107	0.0536	0.00536	0.2676	0.02676
$Q_{7,i}$ (m^3 h^{-1})	945	0	162	0	216	21.6	0	0	729	0
$Q_{i,7}$ (m^3 h^{-1})	945	0	162	0	216	21.6	364.5	0	0	0

Supply (cell 7) air concentration C_7 (ppm) = 300.
Double door flows $Q_{i,j}$ (m^3 h^{-1}) : mean = 216. SD = 21.6.
Single door flows $Q_{i,j}$ (m^3 h^{-1}) : mean = 108 ($Q_{6,4} = Q_{6,5} = 472.5$). SD = 10.8.

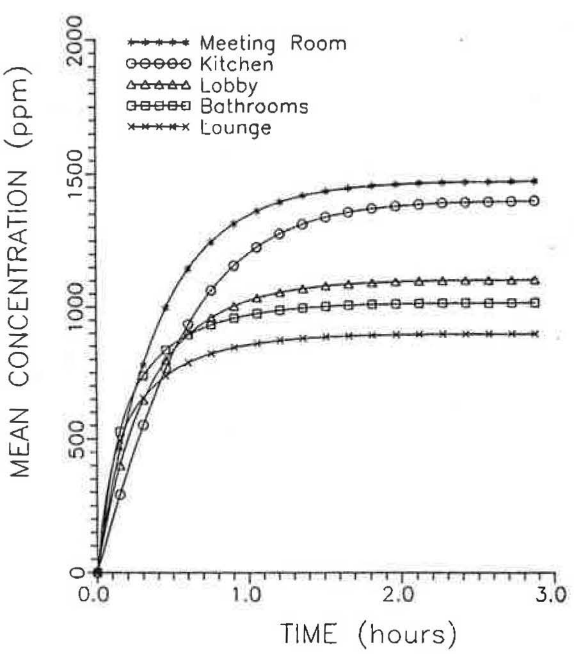

Fig. 7. Mean carbon dioxide concentrations.

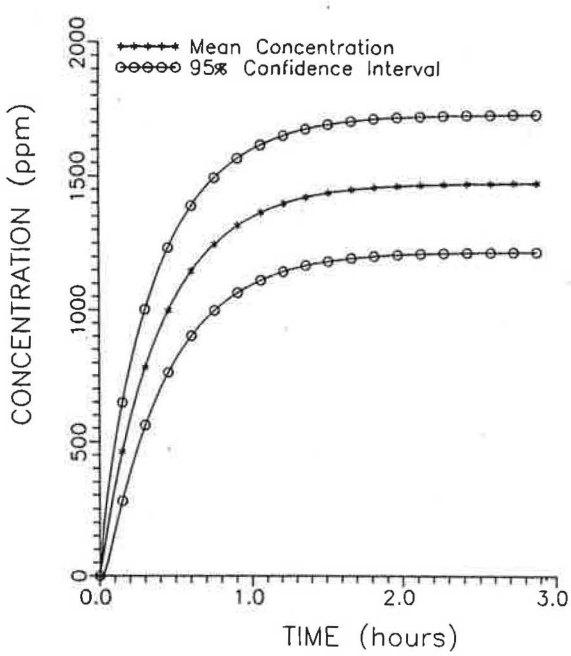

Fig. 9. 95% confidence interval of hall carbon dioxide concentration.

The mean concentration of carbon dioxide in the lounge and the 95% confidence intervals for cases A and B are shown in Fig. 10. The confidence interval widens for case B due to the higher uncertainty of contaminant production in the hall. The ratio of the standard deviation to the mean concentration of carbon dioxide in the lounge increases from 13% to 17%, due to a corresponding increase of from 10% to 33.3% in the ratio of the mean to the standard deviation of contaminant production in the hall. This indicates a relatively insensitive response of uncertainty in contaminant concentration in the lounge to changes in the uncertainty level of contaminant production in the hall.

CONCLUSION

A generalized N-cell, multi-chamber SDE ventilation model with a numerical solution has been developed for the Itô and Stratonovich interpretations of the stochastic integral. This model provides a mathematical tool which

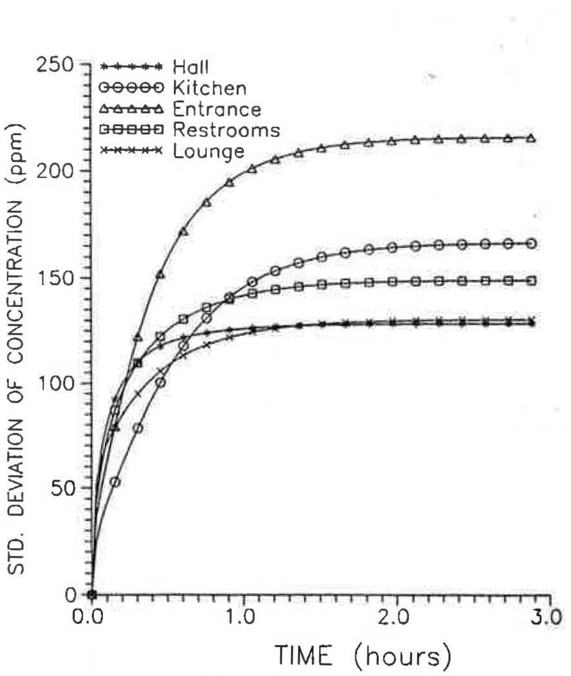

Fig. 8. Standard deviation of carbon dioxide concentrations.

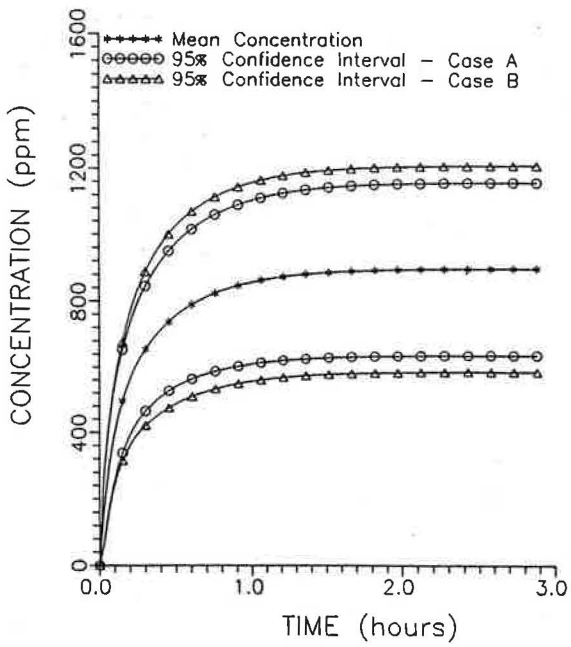

Fig. 10. 95% confidence intervals of lounge carbon dioxide concentration.

can handle uncertainty when modeling the concentration of an air contaminant in a ventilation system which is described by a number of inter-connected, perfectly mixed cells. These cells typically represent rooms in a building. Initial conditions and ventilation parameters may be described by Gaussian probability distributions and the resulting concentration histories in each cell are described by time dependent means and standard deviations. This enables statements to be made about the probability of a contaminant concentration exceeding some specified level. The effect of changing the mean or standard deviation of model parameters can be investigated in sensitivity studies.

The numerical solution will formulate and solve the governing moment equations, given basic model parameters. This makes application of SDE theory to ventilation problems relatively simple and will facilitate ongoing research to determine how well the model predicts actual ventilation problems. The general nature of the numerical solution also makes it applicable to other engineering problems.

Regarding the characteristics of the SDE model solutions, it was shown that the Itô and Stratonovich solutions do not differ significantly for ventilation systems. With respect to modeling the variability in ventilation parameters with Gaussian white noise, this is a reasonable approximation if the time scale of the variability in the parameters is much smaller than the time scale over which the variation in contaminant concentration is being studied.

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