

Multizone Tracer Gas Infiltration Measurements—Interpretation Algorithms for Non-Isothermal Cases

CLAUDE-A. ROULET*
RAPHAËL COMPAGNON*

The algorithms for interpreting multizone air flow measurements using tracer gases presented in the literature are based on implicit rather than clearly defined hypotheses, an important example being that the temperature of the air is assumed to be homogeneous and constant. Moreover, an error analysis is often omitted and the techniques hitherto presented may be further developed.

This paper presents the necessary hypotheses, extends the interpretation algorithms to include cases where the air temperature is neither homogeneous nor constant, and describes a general method for the error analysis which can be used for any measurement technique.

1. CONSERVATION EQUATIONS

1.1 Presentation of the problem

THE LITERATURE addressing air flow measurements using tracer gases contains a degree of confusion related to the physical principle involved, which can lead to errors when using the presented equations. These equations are based on the conservation of the mass of tracer and the mass of air, but it is often volumes which are measured and volume flows which are expected to be the result. This misunderstanding arises from the range of definitions of the tracer gas concentration: parts per volume or parts per mass, mass of tracer per volume of air or even molarity or partial pressures (these latter two definitions are not used in this topic).

In the original paper of Sinden [1], the conservation equations of the tracer masses are correctly written in mass units but the fact that the concentration needs to be expressed in mass of tracer per volume of air is not specified. Equation (2) of Sinden, however, expresses a conservation of the volume of air, which is true only when the density is uniform and constant, but this fact is not stated. In the review papers [2, 3], the units used are not clearly defined and the equations are written in volume units whereas the basic principle is the conservation of mass. In [4] and [5], the equations express clearly the conservation of the volume of air, but the hypothesis of constant and uniform density is not stated.

The objective of this paper is to clearly present the units used for the physical quantities, the hypotheses together with the resulting equations, which are modified to be used in the case of varying air density (that is varying air temperature).

1.2 Definitions

There are N zones (enumerated by i and j going from

1 to N) into which N different tracers enumerated by the index k are injected. In principle, each zone receives only one tracer. No tracer is injected in the outside air (zone 0), which is supposed to be of infinite volume.

Let:

- V_i be the volume of zone i (m^3)
- m_i the mass of air in zone i (kg)
- T_i the absolute temperature in zone i (K)
- V_i^k the volume of tracer k in zone i (m^3)
- m_i^k the mass of tracer k in zone i (kg)
- c_i^k the volume concentration of tracer k in zone i (m^3/m^3)
- C_i^k the mass concentration of tracer k in zone i (kg/kg)
- q_i^k the volume injection rate of tracer k in zone i ($m^3 s^{-1}$)
- Q_i^j the volume air flow rate from zone i to zone j ($m^3 s^{-1}$)
- ρ_i the density of the air in zone i : $\rho_i = m_i/V_i$ ($kg m^{-3}$)
- ρ_i^k the density of tracer k in zone i : $\rho_i^k = m_i^k/V_i^k$ ($kg m^{-3}$)
- ρ^k the density of tracer k when injected ($kg m^{-3}$)
- M^k the molecular mass of tracer k ($kg mol^{-1}$)
- δ_{ij} the "non-Kronecker" delta: $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$.

All variables, except M^k and δ_{ij} , are functions of the time.

1.3 Basic hypotheses

(H1) In each zone, the tracer concentrations are always homogeneous.

(H2) The atmospheric pressure is constant and homogeneous.

(H3) The injection of tracer does not change the density of the air.

* Laboratoire d'Énergie Solaire, Ecole Polytechnique Fédérale, CH-1015 Lausanne, Switzerland.

The first hypothesis is the weakest. In practice, a homogeneous concentration can only be achieved by the use of mixing fans, but these fans have the effect of changing the temperature gradients and therefore the infiltration conditions.

The other two hypotheses are easily satisfied, since the short time variations of the atmospheric pressure are of the order of 10^{-4} (daily variations of the order of 1%) and the tracer gases are generally injected at very low concentrations (10^{-4} in volume or less).

1.4 Conservation of the mass of tracer gas k in zone i

The mass concentration is, assuming that $m_i^k \ll m_i$ (H3):

$$C_i^k \approx \frac{m_i^k}{\rho_i V_i} = \frac{\rho_i^k V_i^k}{\rho_i V_i} \quad \text{hence} \quad m_i^k \approx \rho_i V_i C_i^k. \quad (1)$$

The conservation equation of the mass of tracer k in zone i states that the change of tracer mass within the zone is the sum of the mass of injected tracer and the mass of tracer contained in the air coming into the zone, minus the mass of tracer contained in the outgoing air:

$$\frac{d}{dt}(\rho_i V_i C_i^k) = \rho_i^k q_i^k + \sum_{j=0}^N \rho_j C_j^k Q_{ji} \delta_{ij} - \rho_i C_i^k \sum_{j=0}^N Q_{ij} \delta_{ij}. \quad (2)$$

Mass change
Injection
Infiltration
Exfiltration

An extension of the hypothesis (H1) is implicit in this equation.

(H4) The air flows entering a zone do not modify the homogeneity of the concentration in that zone. In other words, an immediate and perfect mixing is assumed.

If we replace in equation (2) the mass concentration by the volume concentrations given by:

$$c_i^k = \frac{V_i^k}{V_i} \quad \text{hence} \quad C_i^k = \frac{\rho_i^k}{\rho_i} c_i^k, \quad (3)$$

we obtain:

$$\frac{d}{dt}(\rho_i V_i c_i^k) = \rho_i^k q_i^k + \sum_{j=0}^N \rho_j^k c_j^k Q_{ji} \delta_{ij} - \rho_i^k c_i^k \sum_{j=0}^N Q_{ij} \delta_{ij}. \quad (4)$$

The tracer density is defined by $\rho_i^k = m_i^k / V_i^k$ where the volume V_i^k is at the atmospheric pressure p . Using the ideal gas law for the tracer k :

$$p V_i^k = m_i^k R T_i / M^k \quad (5)$$

(where R is the molar gas constant (8.31396×10^{-23} J K^{-1}), M^k the molar mass of the tracer k and T_i the absolute temperature of zone i), the density can be computed:

$$\rho_i^k = \frac{p M^k}{R T_i}, \quad (6)$$

and substituted in equation (4). Dividing this equation by $p M^k$, we get:

$$\frac{d}{dt} \left[\frac{V_i c_i^k}{T_i} \right] = \frac{q_i^k}{T} + \sum_{j=0}^N \frac{c_j^k Q_{ji} \delta_{ij}}{T_j} - \frac{c_i^k}{T_i} \sum_{j=0}^N Q_{ij} \delta_{ij}. \quad (7)$$

The left hand side of this equation can be expanded, for a zone with constant volume:

$$\frac{d}{dt} \left[\frac{V_i c_i^k}{T_i} \right] = \frac{V_i}{T_i} \dot{c}_i^k - \frac{V_i c_i^k}{T_i^2} \dot{T}_i. \quad (8)$$

Equations (7) and (8) represent a complete statement of the conservation of the mass of tracer gas. To derive the form encountered in the literature from these equations, we need two more hypotheses.

(H5) The temperature does not vary with the time.

(H6) The temperature is the same throughout the whole building and the outside air.

Hypothesis (H5) is more important than it appears. In fact, this hypothesis wants to make the second term of the right hand side of equation (8) negligible when compared to the first one. In this first term, there is the derivative of the concentration, which is in principle zero if the constant concentration technique is used. In this case, even very small temperature variations might be too large to satisfy hypothesis (H5).

In fact, hypothesis (H5) can be replaced by the hypothesis that the relative variations of the temperature in zone i is negligible when compared to the relative variations of the concentration of any tracer. That is:

$$\dot{T}_i / T_i \ll \dot{c}_i^k / c_i^k.$$

Hypothesis (H6) would be satisfied if the temperatures of the various zones are the same, if the tracer is injected at the building temperature and finally if the measured building is neither heated or cooled. Otherwise, relative temperature or density differences of up to 10% might be reached (say 30 K indoor-outdoor temperature difference for an indoor temperature of 293 K).

Applying both hypotheses (H5) and (H6) to equation (7), we get the usual form encountered in the literature:

$$V_i \dot{c}_i^k = q_i^k + \sum_{j=0}^N c_j^k Q_{ji} \delta_{ij} - c_i^k \sum_{j=0}^N Q_{ij} \delta_{ij}. \quad (9)$$

1.5 Air mass conservation

The conservation equation for the air mass is obtained the same way as equation (7), except that here the tracer is the air itself, which means that: (i) there is no air injection flowrate ($q_i^k = 0$), (ii) concentrations are all unity $c_i^k = 1$, hypothesis (H3). Applying all the hypotheses (H1) through (H4), we get:

$$\frac{d}{dt}(\rho_i V_i) = \sum_{j=0}^N \rho_j Q_{ji} \delta_{ij} - \rho_i \sum_{j=0}^N Q_{ij} \delta_{ij}, \quad (10)$$

or, using again the perfect gas law:

$$-\frac{V_i}{T_i^2} \dot{T}_i = \sum_{j=0}^N \frac{Q_{ji} \delta_{ij}}{T_j} - \frac{1}{T_i} \sum_{j=0}^N Q_{ij} \delta_{ij}. \quad (11)$$

If all the internal temperatures ($i, j > 0$) are assumed to be equal but different from the outdoor temperature T_0 , we get:

$$\frac{\rho_0}{\rho_i} Q_{0i} + \sum_{j=1}^N Q_{ji} \delta_{ij} = \sum_{j=0}^N Q_{ij} \delta_{ij} + V_i \frac{\dot{\rho}_i}{\rho_i}, \quad (12)$$

where ρ_i is the indoor air density. With the ideal gas law, we get:

$$\frac{T_i}{T_0} Q_{0i} + \sum_{j=1}^N Q_{ji} \delta_{ij} = \sum_{j=0}^N Q_{ij} \delta_{ij} - V_i \frac{\dot{T}_i}{T_i} \quad (13)$$

Here again, the air mass conservation equation found in the literature implies that the indoor temperature is constant and equal to the outdoor temperature.

1.6 Basic equations

Substituting relation (11) in equation (7) combined with (8), we obtain N^2 equations for N^2 unknowns Q_{ji} , which are limited only by the hypotheses (H1)–(H4):

$$\frac{V_i}{T_i} \dot{c}_i^k - \frac{V_i c_i^k}{T_i^2} \dot{T}_i = \frac{q_i^k}{T} + \sum_{j=0}^N \frac{c_j^k Q_{ji} \delta_{ij}}{T_j} - c_i^k \left[\sum_{j=0}^N \frac{Q_{ji} \delta_{ij}}{T_j} + \frac{V_i}{T_i^2} \dot{T}_i \right] \quad (14)$$

This system of equations can be rearranged and coupled with the system (11) to get $N(N+1)$ equations for $N(N+1)$ unknowns. This system allows us (in principle) to compute all the air flow rates at any time from the measured values of the tracers concentrations, the injection rates and the temperatures.

$$\frac{V_i}{T_i} \dot{c}_i^k = \frac{q_i^k}{T} + \sum_{j=0}^N \frac{(c_j^k - c_i^k)}{T_j} Q_{ji} \delta_{ij} \quad (15)$$

$$Q_{i0} = T_i \sum_{j=0}^N \frac{Q_{ji} \delta_{ij}}{T_j} - \sum_{j=1}^N Q_{ij} \delta_{ij} + \frac{V_i}{T_i} \dot{T}_i \quad (16)$$

In equations (15) and (16), as everywhere else in the paper, the temperature of the volume flow Q_{ji} is the temperature T_j of zone j , from which the flow is coming.

If the hypotheses (H5) and (H6) are satisfied, the usual and simpler form can be used.

$$V_i \dot{c}_i^k = q_i^k + \sum_{j=0}^N (c_j^k - c_i^k) Q_{ji} \delta_{ij} \quad (17)$$

$$Q_{i0} = \sum_{j=0}^N Q_{ji} \delta_{ij} - \sum_{j=1}^N Q_{ij} \delta_{ij} \quad (18)$$

Note that for a single zone ($i = 1, j = 0$), equations (15) and (16) simplify to:

$$\frac{V_1}{T_1} \dot{c}_1 = \frac{q_1}{T} + \frac{(c_0 - c_1)}{T_0} Q_{01} \quad (19)$$

and

$$Q_{10} = \frac{T_1}{T_0} Q_{01} + \frac{V_1}{T_1} \dot{T}_1$$

and, assuming constant and homogeneous temperature, equations (17) and (18) become:

$$V_1 \dot{c}_1 = q_1 + (c_0 - c_1) Q_{01} \quad \text{and} \quad Q_{10} = Q_{01}$$

2. COMPUTATION OF AIR FLOWS FROM THE MEASUREMENTS

2.1 Zone-by-zone systems of equations

At each measurement time interval, the measurements give discrete values of c_i^k, T_i and q_i^k ($i, k = 1$ to N). From

these, the flows Q_{ij} can be computed. Here, i and j vary from 0 to N but $i \neq j$.

In equation (15) or (17), i and k vary independently from 1 to N . There is hence a system of N^2 equations which, when combined with N versions of equations (16) or (18), allows us to find the $N^2 + N$ unknowns Q_{ij} .

The systems of equations (15) and (17) are an assembly of N independent sub-systems containing the equations for each given zone. Each sub-system can be written in a matrix form:

$$Y_i = C_i' X_i \quad (20)$$

where Y_i is the vector having N components Y_i^k :

$$Y_i^k = \frac{V_i}{T_i} \dot{c}_i^k - \frac{q_i^k}{T} \quad (21)$$

(where k runs from 1 to N) or, if hypotheses (H5) and (H6) are assumed:

$$Y_i = (V_i \dot{c}_i^1 - q_i^1, \dots, V_i \dot{c}_i^N - q_i^N) \quad (22)$$

Note that in principle, only q_i^i differs from zero, since only gas i is injected in zone i . Moreover, if the injection is perfect in a constant concentration technique, $\dot{c}_i^i = 0$.

X_i' is the vector containing the unknown flows to the zone i :

$$X_i' = (Q_{0i}, Q_{1i}, \dots, Q_{i-1,i}, 0, Q_{i+1,i}, \dots, Q_{Ni}), \quad (23)$$

and C_i' a matrix having the element δc_{kj} in row k and column j :

$$\delta c_{kj} = \frac{c_j^k - c_i^k}{T_j} \quad (24)$$

or, assuming hypotheses (H5) and (H6):

$$\delta c_{kj} = c_j^k - c_i^k \quad (25)$$

The matrix C_i' has N rows and $N+1$ columns, but the column i is filled with zeroes. The vector X_i' has also $N+1$ components, the component i , corresponding to Q_{ii} being zero. We can therefore contract the system (20) into:

$$Y_i = C_i X_i \quad (26)$$

where X_i is the vector X_i' without component i , and C_i matrix C_i' without column i . The system (26) is then entirely determined if the matrix C_i is not singular. It is solved in principle by:

$$X_i = C_i^{-1} Y_i \quad (27)$$

which gives the N flows Q_{ji} from zones j ($j = 0 \dots N$) to zone i . Solving (26) for the N zones, we get all the flows Q_{ji} where $j = 0 \dots N$ and $i = 1 \dots N$. The N remaining flows Q_{i0} are computed using N versions of equations (16) or (18).

Note that this resolution method differs slightly from the one presented by Sinden [1] and Perera [2] but seems simpler if the constant concentration technique is used.

2.2 Discretization

The time derivatives of the tracer concentration appear in the vector Y_i , and the derivatives of the temperatures are used in equation (16). Equations (26) and (16) can be solved at each time interval, if we write either the backwards derivative:

$$\dot{c}_i^k(t) = [c_i^k(t) - c_i^k(t - \delta t)] / \delta t \quad (28)$$

$$\dot{T}_i^k(t) = [T_i^k(t) - T_i^k(t - \delta t)]/\delta t, \quad (29)$$

or the forwards derivative :

$$\dot{c}_i^k(t) = [c_i^k(t + \delta t) - c_i^k(t)]/\delta t, \quad (30)$$

$$T_i^k(t) = [T_i^k(t + \delta t) - T_i^k(t)]/\delta t. \quad (31)$$

For causality reasons, equations (28) and (29) may be better. A test is running with both possibilities on a set of measurements made at the LESO on a single cell. For this cell, using equation (19) :

$$Q_{01} = \frac{T_0}{c_0 - c_1} \left[\frac{V_1}{T_1} \dot{c}_1 - \frac{q}{T} \right], \quad (32)$$

where T is the temperature of the tracer when injected, T_1 the temperature of the air in the zone and T_0 the outdoor temperature.

In reality, the measurements with the constant concentration techniques are done the following way :

At a time t_0 , a sample of the air of a defined zone is taken and analysed and this operation lasts for the time δt_a .

At time $t_0 + \delta t_a$, the amount of tracer gas necessary to maintain a constant concentration is injected in the zone by opening a valve for a calculated time δt_{inj} .

The start of the next sampling takes part some time after the end of this injection.

Hence, the sampling-analysis-injection cycles lasts a period of time which should be taken as a "natural" time step for the discretization. Usually this time step is also the time interval between the records of measured data.

In this paper, all the variables (except the volumes of the zones) may depend on time but it is implicitly admitted that they are constant during the discretization time step. This assumption seems obvious, since we have no information of the variation of the concentrations between the measurements. Nevertheless, it shall be remembered that the time step may be large (e.g. 10 min) when compared to the time constants of the possible changes of the air and tracer injection flows. Because of this, interpretation errors may occur, generally overestimating the air flow rates by several percent.

It is possible that a more refined interpretation method could be found, taking into account that the tracer injection rate is not constant during the measurement interval and that the tracer concentration may change during that interval. An indication of how this might be done can be found in the work Aittomäkki did for U-value measurements [6].

3. ERROR ANALYSIS

3.1 The problem

Experimentation results in a system of equations to be solved :

$$A\mathbf{x} = \mathbf{y}, \quad (33)$$

where vector \mathbf{y} and matrix A both contain measured coefficients (y_j and a_{ij}) with errors represented respectively by a vector $\delta\mathbf{y}$ and a matrix δA . The question is: which is the resulting error $\delta\mathbf{x}$ on the vector \mathbf{x} ?

Since components x_i are functions of the coefficients a_{ij} and y_j , their variances can be computed approximately

by :

$$s^2(x_i) = \sum_j \frac{\delta x_i}{\delta a_{ij}} s^2(a_{ij}) + \sum_j \frac{\delta x_i}{\delta y_j} s^2(y_j), \quad (34)$$

where $s^2(a_{ij})$ and $s^2(y_j)$ are the variances of the measured coefficients. If the errors in the variables a_{ij} and y_j are independent and symmetrically distributed and if the function is not far from linear, this first order error analysis gives satisfactory results [7]. However the computation time required increases rapidly with the number of variables (that is the number of zones).

If the matrix δA and the vector $\delta\mathbf{y}$ were exactly known for a given case, we could write :

$$(A + \delta A)(\mathbf{x} + \delta\mathbf{x}) = \mathbf{y} + \delta\mathbf{y}, \quad (35)$$

and, taking equation (33) into account, this could be solved :

$$\delta\mathbf{x} = (A + \delta A)^{-1}(\delta A\mathbf{x} + \delta\mathbf{y}). \quad (36)$$

Equation (36) can however not be used to determine the absolute error or a statistical standard deviation. For that latter purpose, we can use equation (36) several times in a Monte-Carlo process, varying the components of δA and $\delta\mathbf{y}$ at random but according to their probability density function. This provides several vectors $\delta\mathbf{x}$ from which an estimate of the probability density functions of the components can be calculated.

Vector $\delta\mathbf{x}$ contains a large number of values. It would be helpful to represent the error by a single figure. To obtain such a single figure, we need the following definitions, which can be found in the specific mathematical literature (e.g. [7, 8]).

3.2 Vectorial and matrix norms

The norm $\|\mathbf{x}\|$ of a vector \mathbf{x} is any operation of R^n in R satisfying :

$$\begin{aligned} \|\mathbf{x}\| &\geq 0 \text{ and } \|\mathbf{x}\| = 0 \text{ if and only if } \mathbf{x} = \mathbf{0} \\ \|\mathbf{c}\mathbf{x}\| &= |c| \|\mathbf{x}\| \text{ for any } c \in R \\ \|\mathbf{x} + \mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\|. \end{aligned} \quad (37)$$

For example, the euclidian norm (which corresponds to the standard deviation if the average $\langle \mathbf{x} \rangle = \mathbf{0}$) :

$$\|\mathbf{x}\|_2 = \sqrt{(\sum x_i^2)}, \quad (38)$$

fulfills the relations (37), but there are other norms, like $\|\mathbf{x}\|_1 = \sum |x_i|$ or the infinity norm defined by $\|\mathbf{x}\|_\infty = \max |x_i|$.

The norm $\|A\|$ of a matrix A is any application $N(A) \rightarrow \|A\| \in R$ satisfying :

$$\begin{aligned} \|A\| &\geq 0 \text{ and } \|A\| = 0 \text{ if and only if } A = \mathbf{0}, \\ \|cA\| &= |c| \|A\| \text{ for any } c \in R, \\ \|A + B\| &\leq \|A\| + \|B\|, \\ \|A \cdot B\| &\leq \|A\| \cdot \|B\|. \end{aligned} \quad (39)$$

The matrix norm $\|A\|$ is *consistent* with the vectorial norm $\|\mathbf{x}\|$ if :

$$\|A \cdot \mathbf{x}\| \leq \|A\| \cdot \|\mathbf{x}\| \text{ for any value of } \mathbf{x}. \quad (40)$$

The matrix norm *subordinated* to the vectorial norm $\|\mathbf{x}\|$ is defined by :

$$\|A\| = \max (\|A\mathbf{x}\|/\|\mathbf{x}\|) \text{ for any } \mathbf{x} \neq \mathbf{0}. \quad (41)$$

The subordinated matrix norm is the smallest one which is consistent with $\|x\|$. For example, the normal $\|A\|_2$, defined by:

$$\|A\|_2 = \sqrt{\mu_1}, \quad (42)$$

where μ_1 is the largest eigenvalue of $A^H A$ (A^H = hermitic conjugate or the transpose of the complex conjugate matrix) and is subordinated to the euclidean norm $\|x\|_2$. This matrix norm is the *spectral norm*.

Both Walker [9] and D'Ottavio [10] present an error analysis method similar to the following one, but they use other matrix and vectorial norms. Walker uses the vectorial infinity norm and the subordinated matrix norm which is the maximum row sum of the moduli of the elements. These norms lead to simpler computations but do not give a good image of the usual standard deviation. D'Ottavio *et al.* [10] use the Frobenius norm:

$$\|A\|_F = \left[\sum_i \sum_j (a_{ij})^2 \right]^{1/2}, \quad (43)$$

which is consistent with the euclidian vectorial norm but which is not subordinated to it, hence not the smallest. Therefore these authors have introduced a correcting factor adjusted to fit the results of this error analysis method to the results of the first order approximation method [equation (34)].

It is proposed here to use the *euclidean vectorial norm* and the subordinated *spectral matrix norm* for the error analysis. The advantages are that the euclidian norm of the vector δy represents its total standard deviation and, using the subordinated matrix norm, we will obtain the smallest upper bound for δx .

3.3 Error analysis

It is possible to give an upper limit to the norm of the resulting error $\|\delta x\|$, using the following relations [7] for the norms of the experimental errors $\|\delta y\|$ and $\|\delta A\|$.

Only y is perturbed. We can write:

$$A(x + \delta x) = y + \delta y, \quad (44)$$

but, taking equation (21) into account, we get:

$$A \delta x = \delta y. \quad (45)$$

Then, for any pair of vectorial and matrix subordinated norms:

$$\|\delta x\| \leq \|A^{-1} \delta y\| \leq \|A^{-1}\| \cdot \|\delta y\|, \quad (46)$$

and:

$$\frac{\|\delta x\|}{\|x\|} \leq \|A\| \cdot \|A^{-1}\| \cdot \frac{\|\delta y\|}{\|y\|}.$$

The number:

$$\text{cond}(A) = \|A\| \cdot \|A^{-1}\|, \quad (47)$$

is of great importance here. It is the *condition number* of the matrix A related to the used norm. If $\|A\|_2$ is used, we get the smallest possible condition number. Its value is:

$$\text{cond}_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2 = \sqrt{(\mu_1/\mu_n)}, \quad (48)$$

where μ_1 and μ_n are respectively the largest and the smallest eigenvalues of $A^{H1} \cdot A$. This condition number is the spectral condition number.

Only matrix A is perturbed. In this case:

$$(A + \delta A)(x + \delta x) = y \text{ then } (A + \delta A)\delta x = -\delta A x \quad (49)$$

If $(A + \delta A)$ is regular, hence if $\|A^{-1} \delta A\| \leq \|A^{-1}\| \cdot \|\delta A\| < 1$, then:

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1} \delta A\|}{1 - \|A^{-1} \delta A\|} \leq \frac{\|A\| \cdot \|A^{-1}\| \cdot \|\delta A\| / \|A\|}{1 - \|A\| \cdot \|A^{-1}\| \cdot \|\delta A\| / \|A\|}. \quad (50)$$

A and b are perturbed. In this case, assuming that $\|I\| = 1$ (which is true for $\|I\|_2$):

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A\| \cdot \|A^{-1}\|}{1 - \|\delta A\| \cdot \|A^{-1}\|} \cdot \left[\frac{\|\delta y\|}{\|y\|} + \frac{\|\delta A\|}{\|A\|} \right]. \quad (51)$$

3.4 How to obtain a good condition

To minimize the condition number of the experimental matrix, some variable changes can be used on the initial problem in such a way that the coefficients a_{ij} of the matrix A satisfy:

$$1/\beta \leq \max(|a_{ij}|) \leq 1, \quad (52)$$

where β is the base of the floating point arithmetic system installed in the computer used to solve the system of equations. For that purpose, it is often enough to divide each row of the system by the largest coefficient, rounded to an integer power of β , to minimize the rounding errors.

Moreover, in the Gaussian resolution procedure, the equations are permuted in such a way that the largest (in absolute value) pivot is used.

3.5 Error analysis for two special cases

If e and ϵ are the constant relative errors on A and y :

$$\delta A = eA \text{ and } \delta y = \epsilon y, \quad (53)$$

and, from the definition of the norms:

$$\|\delta A\| = e\|A\| \text{ and } \|\delta y\| = \epsilon y. \quad (54)$$

We have, for any definition of the norm satisfying $\|I\| = 1$:

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\text{cond}(A)}{1 - e \text{cond}(A)} \cdot (\epsilon + e). \quad (55)$$

It should be remembered that $A + \delta A$ should be regular, which will occur if:

$$e \cdot \text{cond}(A) < 1. \quad (56)$$

In the case of a constant absolute error:

$$\delta A = eI \text{ and } \delta y = \delta y \mathbf{1}, \quad (57)$$

where I and $\mathbf{1}$ are respectively a matrix of order N and a vector with N components where all the elements are unity.

It is easily seen that, if the euclidian and spectral norms are used:

$$\|I\|_2 = N \text{ and } \|\mathbf{1}\|_2 = \sqrt{N}. \quad (58)$$

In fact, the eigenvalues of I are N and 0^{N-1} , hence those of $I^H I = I^2$ are N^2 and 0^{N-1} . Then:

$$\|\delta A\|_2 = eN \text{ and } \|\delta y\|_2 = \sqrt{N} \delta y \quad (59)$$

and:

$$\frac{\|\delta x\|_2}{\|x\|_2} \leq \frac{\text{cond}_2(A)}{1 - eN \cdot \|A^{-1}\|_2} \cdot \left[\frac{\sqrt{N} \delta y}{\|y\|_2} + \frac{Ne}{\|A\|_2} \right] \quad (60)$$

We see that for these simple cases, which are the most common in practice, an upper bound for the relative error on the results can be easily calculated when the condition number of the matrix A is known. This matrix is C , in equation (26).

4. AN EXAMPLE

4.1 Case configuration

A constant concentration measurement on a very simple building with two rooms was simulated using a multizone infiltration computer code to provide the data used in this example. This way, the measurement errors are avoided. The input data are given in Table 1.

The volume flows Q_{j0} to the outside result from the mass conservation of the air. After a few hours, the tracer concentrations stabilize and the results of the simulated measurements are those given in Table 2.

4.1 Results for uniform temperature

Using equation (17) for zone 1, we get:

$$\begin{bmatrix} -2.65 \times 10^{-4} \\ 0 \end{bmatrix} = \begin{bmatrix} -10 & -7.71 \\ -6.46 & 3.54 \end{bmatrix} 10^{-6} \begin{bmatrix} Q_{01} \\ Q_{21} \end{bmatrix}$$

Table 1.

	Zone 0 (Outside)	Zone 1	Zone 2	
Temperature	0	20	20	°C
Volume	—	50	100	m ³
Flows from outside Q_{0i}	—	10	30	m ³ h ⁻¹
Flows from zone 1 Q_{1i}	20.73	—	10	m ³ h ⁻¹
Flows from zone 2 Q_{2i}	22.2	20	—	m ³ h ⁻¹

which gives the flows:

$$Q_{01} = 10.99 \text{ m}^3 \text{ h}^{-1} \quad \text{and} \quad Q_{21} = 20.06 \text{ m}^3 \text{ h}^{-1}$$

The same calculations for zone 2 gives:

$$Q_{02} = 32.57 \text{ m}^3 \text{ h}^{-1} \quad \text{and} \quad Q_{12} = 9.68 \text{ m}^3 \text{ h}^{-1}$$

Finally, using equation (18) we get:

$$Q_{10} = 21.37 \text{ m}^3 \text{ h}^{-1} \quad \text{and} \quad Q_{20} = 22.19 \text{ m}^3 \text{ h}^{-1}$$

Comparing these results with the exact values given above, we see that the largest differences, of about 10%, occur for Q_{01} and Q_{02} , which are the flows coming from the cold.

4.3 Results taking into account the temperature

Using the new proposed equations (15) and (16), the following results are obtained:

$$Q_{01} = 10.24 \text{ m}^3 \text{ h}^{-1} \quad Q_{21} = 20.06 \text{ m}^3 \text{ h}^{-1}$$

$$Q_{02} = 30.35 \text{ m}^3 \text{ h}^{-1} \quad Q_{12} = 9.68 \text{ m}^3 \text{ h}^{-1}$$

$$Q_{10} = 21.37 \text{ m}^3 \text{ h}^{-1} \quad Q_{20} = 22.19 \text{ m}^3 \text{ h}^{-1}$$

Since both zones have the same temperatures, only the flows coming from outside are changed and these results are more accurate than the results found in Section 4.2. The remaining differences (3% or less) are caused by rounding errors, since only three digits were kept for the simulated concentrations and tracer injection flow rates.

4.4 Error analysis

The matrices of equation (15) were reconditioned (Section 3.4) before solving the system, in such a way that the largest elements (in absolute value) of each row were equal to 1. A uniform relative error was assumed, being 2% on the concentration matrix and 5% on the injection vector (that is $e = 0.02$ and $\varepsilon = 0.05$). For this reconditioned system of equations and using two different norms, the results given in Table 3 for $\|\delta Q\|/\|Q\|$ were obtained using equation (55).

We see that the spectral norm gives a smaller upper bound for the relative error than the Frobenius norm,

Table 2.

	Zone 1	Zone 2	
Concentration of tracer 1	10	2.29	ppm
Concentration of tracer 2	6.46	10	ppm
Injection rate of tracer 1	2.65×10^{-4}	0	m ³ h ⁻¹
Injection rate of tracer 2	0	3.6×10^{-4}	m ³ h ⁻¹

Table 3.

Norm used	Frobenius [equation (43)]		Spectral [equation (42)]	
	1	2	1	2
Zone				
Matrix norm $\ A\ $	1.666	1.487	1.426	1.057
and for the inverse $\ A^{-1}\ $	1.356	1.355	1.161	0.956
$\text{cond}(A) = \ A\ \cdot \ A^{-1}\ $	2.259	2.015	1.656	1.010
Relative error $\ \delta Q\ /\ Q\ $	0.166	0.146	0.120	0.072

the difference going up to a factor of two for the second zone.

If the matrix were not reconditioned, we would get, for the zone 1 and with the spectral norm:

$$\|\delta Q\|/\|Q\| = 0.153$$

which is only slightly more than the conditioned matrix in this case, since the system is already not too badly conditioned. For badly conditioned systems, this difference may be much larger.

5. CONCLUSIONS

In this paper, we have proposed modification of the equations of conservation of the tracer gases and of the

air in such a way that measurements made with variable and non homogeneous temperatures can be interpreted properly.

Secondly, in order to obtain error figures which are not exaggeratedly large and which represent standard deviations, it is proposed to use the euclidian vectorial norm and the subordinated spectral matrix norm.

Acknowledgements—This work is done within the framework of a coordinated Swiss research program on air flow patterns within buildings. It is sponsored by the Swiss national energy research fund (NEFF) with the credit 339.2. The authors thank Peter Lewis from the University of Cardiff for revision of the manuscript.

REFERENCES

1. F. W. Sinden, Multi-chamber theory of air infiltration. *Bldg Envir.* **13**, 21–28 (1978).
2. M. D. A. E. S. Perrera, Review of techniques for measuring ventilation rates in multi-celled buildings. Proceedings of the EC Contractor's meeting on Natural Ventilation, Bruxelles (1982).
3. P. S. Charlesworth, *A Guide to Air Exchange Rates and Airtightness Measurement Techniques*. Air Infiltration and Ventilation Center, Warwick (1988).
4. M. H. Sherman, D. T. Grimsrud, P. E. Condon and B. V. Smith, Air infiltration measurement techniques. Proceedings of the first AIC Conference on air infiltration instrumentation and measuring techniques (1980).
5. M. Sandberg, The multi-chamber theory reconsidered from the viewpoint of air quality studies. *Bldg Envir.* **19**, 221–233 (1984).
6. S. Ahvenainen, E. Kokko and A. Aittomäkki, Thermal conductances of wall structures. Laboratory of Heating and Ventilating, Technical Research Centre of Finland, Espoo, Report No. 54 (1980).
7. P. Arminjon, *Analyse Numerique Matricielle*. Presses de l'Université de Montréal (1978).
8. A. Deif, *Sensitivity Analysis in Linear Systems*. Springer, Berlin (1986).
9. R. R. Walker, Interpretation and error analysis of multi-tracer gas measurement to determine air movements in a house. Proceedings of the 6th AIC Conference on Ventilation strategies and measurement techniques (1985).
10. T. W. D'Ottavio, G. I. Senum and R. N. Dietz, Error analysis techniques for perfluorocarbon tracer derived multizone ventilation rates. *Bldg Envir.* **23**, 187–194 (1988).