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MULTI-ZONE TRACER GAS INFILTRATION MEASUREMENTS INTERPRETATION ALGORITHMS FOR NON ISOTHERMAL CASES

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ABSTRACT

The algorithms for interpreting multizone air flow measurements using tracer gases which are presented in the literature are based on implicit rather than clearly defined hypotheses, an important example being that the temperature of the air is assumed to be homogeneous and constant. Moreover, an error analysis is often omitted and the techniques presented till now can still be developped.

This paper presents the necessary hypotheses, extends the interpretation algorithms to include cases where the air temperature is neither homogeneous nor constant, and describes a general method for the error analysis which can be used for any measurement technique.

1 CONSERVATION EQUATIONS

1.1 Présentation of the problem

The literature addressing air flow measurements using tracer gases contains a degree of confusion related to the physical principle involved, which can lead to errors when using the presented equations. These equations are based on the conservation of the *mass* of tracer and the *mass* of air, but it is often *volumes* which are measured and *volume flows* which are expected to be the result. The misunderstanding arises from the range of definitions of the tracer gas concentration: parts per volume or parts per mass, mass of tracer per volume of air or even molarity or partial pressures (these latter two definitions are not used in this topic).

In the original paper of Sinden [1], the conservation equations of the tracer masses are correctly written in mass units but does not specify that the concentration needs to be expressed in mass of tracer per volume of air. The equation (2) of Sinden, however, expresses a conservation of the volume of air, which is true only when the density is uniform and constant, but this fact is not stated. In the review papers [2 and 3], the units used are not clearly defined and the equations are written in *volume* units whereas the basic principle is the conservation of *mass*. In [4] and [5], the equations express clearly the conservation of the volume of air, but the hypothesis of constant and uniform density is not stated.

The objective of this paper is to present clearly the units used for the physical quantities, the hypotheses together with the resulting equations, which are modified to be used in the case of varying air density (that is varying air temperature)

1.2 Définitions

There are N zones (enumerated by i and j going from 1 to N) into which N different tracers enumerated by the index k are injected. In principle, each zone receives only one tracer. No tracer is injected in the outside air (the zone 0), which is supposed to be of infinite volume.

Let V_i be the volume of the zone i $[m^3]$
* m_i the mass of air in the zone i $[kg]$
* T_i the absolute temperature in the zone i $[K]$
* V_i^k the volume of tracer k in the zone i $[m^3]$
* m_i^k the mass of tracer k in the zone i $[kg]$
* c_i^k the volume concentration of the tracer k in the zone i $[m^3/m^3]$
* C_i^k the mass concentration of the tracer k in the zone i $[kg/kg]$
* q_i^k the volume injection rate of the tracer k in the zone i $[m^3/s]$
* Q_{ij} the volume air flow rate from the zone i to zone j $[m^3/s]$
* ρ_i the density of the air in the zone i: $\rho_i = m_i / V_i$ $[kg/m^3]$
* ρ_i^k the density of the tracer k in the zone i: $\rho_i^k = m_i^k / V_i^k$ $[kg/m^3]$
* ρ^k the density of the tracer k when injected $[kg/m^3]$
M^k the molecular mass of the tracer k $[kg/Mole]$
δ_{ij} the "non- Kronecker" delta: $\delta_{ij} = 0$ if $i=j$, $\delta_{ij} = 1$ if $i \neq j$

The variables marked with an asterisk are functions of the time.

1.3 Basic hypotheses

H 1: ***In each zone, the tracer concentrations are always homogeneous.***
H 2: ***The atmospheric pressure is constant and homogeneous.***
H 3: ***The injection of tracer does not change the density of the air.***

The first hypothesis is the weakest. In practice, an homogeneous concentration can only be achieved by the use of mixing fans, but these fans have the effect of changing the temperature gradients and therefore the infiltration conditions.

The other two hypotheses are easily satisfied, since the short time variations of the atmospheric pressure are of the order of 10^{-4} (daily variations of the order of a percent) and the tracer gases are generally injected at very low concentrations (10^{-4} in volume or less).

1.4 Conservation of the mass of the tracer gas k in the zone i

The mass concentration is, assumed that $m_i^k \ll m_i$ (hypothesis H 3):

$$C_i^k \approx \frac{m_i^k}{\rho_i \, V_i} = \frac{\rho_i^k \, V_i^k}{\rho_i \, V_i} \qquad \text{hence} \qquad m_i^k \approx \rho_i \, V_i \, C_i^k \qquad (1)$$

The conservation equation of the mass of tracer k in the zone i states that the change of tracer mass within the zone is the sum of the mass of injected tracer and the mass of tracer contained in the air coming into the zone, minus the mass of tracer contained in the outgoing air:

$$\underbrace{\frac{d}{dt}(\rho_i V_i C_i^k)}_{\text{Mass change}} = \underbrace{\rho^k q_i^k}_{\text{Injection}} + \underbrace{\sum_{j=0}^{N} \rho_j C_j^k Q_{ji} \delta_{ij}}_{\text{infiltration}} - \underbrace{\rho_i C_i^k \sum_{j=0}^{N} Q_{ij} \delta_{ij}}_{\text{Exfiltration}} \qquad (2)$$

An extension of the hypothesis H 1 is implicit in this equation, which is:

H 4: ***The air flows entering a zone do not modify the homogeneity of the concentration in that zone. In other words, an immediate and perfect mixing is assumed.***

If we replace in equation (2) the mass concentration by the volume concentrations given by:

$$c_i^k = \frac{V_i^k}{V_i} \qquad \text{hence} \quad C_i^k = \frac{\rho_i^k}{\rho_i} c_i^k \qquad (3)$$

we obtain:

$$\frac{d}{dt}(\rho_i^k V_i c_i^k) = \rho^k q_i^k + \sum_{j=0}^{N} \rho_j^k c_j^k Q_{ji} \delta_{ij} - \rho_i^k c_i^k \sum_{j=0}^{N} Q_{ij} \delta_{ij} \qquad (4)$$

The tracer density is defined by $\rho_i^k = m_i^k/V_i^k$ where the volume V_i^k is at the atmospheric pressure p. Using the perfect gases law for the tracer k:

$$p V_i^k = m_i^k R T_i / M^k \qquad (5)$$

(where R is the molar gas constant ($8.31396 \cdot 10^{-23}$ J/K), M^k the molar mass of the tracer k and T_i the absolute temperature of the zone i), the density can be computed:

$$\rho_i^k = \frac{p \cdot M^k}{R \cdot T_i} \qquad (6)$$

and substituted in equation (4). Dividing this equation by $p \cdot M^k$, we get:

$$\frac{d}{dt}\left[\frac{V_i c_i^k}{T_i}\right] = \frac{q_i^k}{T} + \sum_{j=0}^{N} \frac{c_j^k Q_{ji} \delta_{ij}}{T_j} - \frac{c_i^k}{T_i} \cdot \sum_{j=0}^{N} Q_{ij} \delta_{ij} \qquad (7)$$

The left hand side of this equation can be expanded, for a zone with constant volume:

$$\frac{d}{dt}\left[\frac{V_i c_i^k}{T_i}\right] = \frac{V_i}{T_i} \dot{c}_i^k - \frac{V_i c_i^k}{T_i^2} \dot{T}_i \qquad (8)$$

Equations (7) and (8) represent a complete statement of the conservation of the mass of tracer gas. To derive the form encountered in the literature from these equations, we need two more hypotheses:

H 5: The temperature does not vary with the time.

H 6: The temperature is the same throughout the whole building and the outside air.

Hypothesis H 5 is more important than it appears. In fact, this hypothesis wants to make the second term of the right hand side of equation (8) negligible when compared to the first one. In this first term, there is the derivative of the concentration, which is in principle zero if the constant concentration technique is used. In this case, even very small temperature variations might be too large to satisfy hypothesis H 5.

In fact, hypothesis H 5 can be replaced by the hypothesis that *the relative variations of the temperature in the zone i is negligible when compared to the relative variations of the concentration of any tracer.* That is:

$$\dot{T}_i / T_i \ll \dot{c}_i^k / c_i^k$$

Hypothesis H 6 would be satisfied if the temperatures of the various zones are the same, if the tracer is injected at the building temperature and finally if the measured building is neither heated or cooled. Otherwise, relative temperature or density differences of up to 10 % might be reached (say 30 K indoor-outdoor temperature difference for an indoor temperature of 293 K).

Applying both hypotheses H 5 and H 6 to equation (7), we get the usual form encountered in the literature:

$$V_i \dot{c}_i^k = q_i^k + \sum_{j=0}^{N} c_j^k Q_{ji} \delta_{ij} - c_i^k \sum_{j=0}^{N} Q_{ij} \delta_{ij} \qquad (9)$$

1.5 Air mass conservation

The conservation equation for the air mass is obtained the same way as equation (7), except that here the tracer is the air itself, which means that:

- there is no air injection flowrate ($q_i^k = 0$)
- concentrations are all unity ($c_i^k = 1$, hypothesis H 3).

Applying all the hypotheses H 1 through H 4, we get:

$$\frac{d}{dt}(\rho_i V_i) = \sum_{j=0}^{N} \rho_j Q_{ji} \delta_{ij} - \rho_i \sum_{j=0}^{N} Q_{ij} \delta_{ij} \qquad (10)$$

or, using again the perfect gas law:

$$-\frac{V_i}{T_i^2} \dot{T}_i = \sum_{j=0}^{N} \frac{Q_{ji} \delta_{ij}}{T_j} - \frac{1}{T_i} \sum_{j=0}^{N} Q_{ij} \delta_{ij} \qquad (11)$$

If all the internal temperatures (i, j > 0) are assumed to be equal but different from the outdoor temperature T_0, we get:

$$\frac{\rho_0}{\rho_i} Q_{0i} + \sum_{j=1}^{N} Q_{ji}\,\delta_{ij} = \sum_{j=0}^{N} Q_{ij}\,\delta_{ij} + V_i\,\frac{\dot{\rho}_i}{\rho_i} \qquad (12)$$

where μ_i is the indoor air density. With the perfect gas law, we get:

$$\frac{T_i}{T_0} Q_{0i} + \sum_{j=1}^{N} Q_{ji}\,\delta_{ij} = \sum_{j=0}^{N} Q_{ij}\,\delta_{ij} - V_i\,\frac{\dot{T}_i}{T_i} \qquad (13)$$

Here again, the air mass conservation equation found in the literature implies that the indoor temperature is constant and equal to the outdoor temperature.

1.6 Basic equations

Substituting relation (11) in equation (7) combined with (8), we obtain N^2 equations for N^2 unknowns Q_{ji}, which are limited only by the hypotheses H 1 through H 4:

$$\frac{V_i}{T_i}\,\dot{c}_i^{\,k} - \frac{V_i\,c_i^{\,k}}{T_i^{\,2}}\,\dot{T}_i = \frac{q_i^{\,k}}{T} + \sum_{j=0}^{N} \frac{c_j^{\,k}\,Q_{ji}\,\delta_{ij}}{T_j} - c_i^{\,k}\left[\sum_{j=0}^{N} \frac{Q_{ji}\,\delta_{ij}}{T_j} + \frac{V_i}{T_i^{\,2}}\,\dot{T}_i\right] \qquad (14)$$

This system of equations can be rearranged and coupled with the system (11) to get N(N+1) equations for N(N+1) unknowns. This system allows us (in principle) to compute all the air flow rates at any time from the measured values of the tracers concentrations, the injection rates and the temperatures:

$$\boxed{\begin{aligned} \frac{V_i}{T_i}\,\dot{c}_i^{\,k} &= \frac{q_i^{\,k}}{T} + \sum_{j=0}^{N} \frac{(c_j^{\,k} - c_i^{\,k})}{T_j}\,Q_{ji}\,\delta_{ij} && (15) \\ Q_{i0} &= T_i \sum_{j=0}^{N} \frac{Q_{ji}\,\delta_{ij}}{T_j} - \sum_{j=1}^{N} Q_{ij}\,\delta_{ij} + \frac{V_i}{T_i}\,\dot{T}_i && (16) \end{aligned}}$$

In equations (15) and (16), as everywhere else in that paper, the temperature of the volume flow Q_{ji} is the temperature T_j of the zone j, from which the flow is coming.

If the hypotheses H 5 and H 6 are satisfied, the usual and simpler form can be used:

$$V_i \dot{c}_i^k = q_i^k + \sum_{j=0}^{N} (c_j^k - c_i^k) Q_{ji} \theta_{ij} \qquad (17)$$

$$Q_{i0} = \sum_{j=0}^{N} Q_{ji} \theta_{ij} - \sum_{j=1}^{N} Q_{ij} \theta_{ij} \qquad (18)$$

Note: For a single zone (i=1, j=0), equations (15) and (16) simplify to:

$$\frac{V_1}{T_1} \dot{c}_1 = \frac{q_1}{T} + \frac{(c_0 - c_1)}{T_0} Q_{01} \quad \text{and} \quad Q_{10} = \frac{T_1}{T_0} Q_{01} + \frac{V_1}{T_1} \dot{T}_1 \qquad (19)$$

and, assuming constant and homogeneous temperature, (17) and (18) become:

$$V_1 \dot{c}_1 = q_1 + (c_0 - c_1) \qquad \text{and} \qquad Q_{10} = Q_{01}$$

2 COMPUTATION OF AIR FLOWS FROM THE MEASUREMENTS.

2.1 Zone by zone systems of equations

At each measurement time interval, the measurements give discrete values of c_i^k, T_i and q_i^k (i, k = 1 to N). From these, the flows Q_{ij} can be computed. Here, i and j vary from 0 to N but $i \neq j$.

In equation (15) or (17), i and k vary independently from 1 to N. There is hence a system of N^2 equations which, combined with the N equations (16) or (18) allow us to find the $N^2 + N$ unknowns Q_{ij}.

The systems of equations (15) and (17) are an assembly of N independent sub-systems containing the equations for each given zone. Each sub-system can be written in a matrix form:

$$\underline{Y_i} = \underline{\underline{C_i}}' \, \underline{X_i}' \qquad (20)$$

where $\underline{Y_i}$ is the vector having N components Y_i^k:

$$Y_i^k = \frac{V_i}{T_i} \dot{c}_i^k - \frac{q_i^k}{T} \quad \text{where k runs from 1 to N} \qquad (21)$$

or, if the hypotheses H 5 and H 6 are assumed:

$$\underline{Y_i} = (V_i \dot{c}_i^1 - q_i^1, \dots, V_i \dot{c}_i^i - q_i^i, \dots, V_i \dot{c}_i^N - q_i^N) \qquad (22)$$

Note: in principle, only q_i^i differs from zero, since only the gas i is injected in the zone i. Moreover, if the injection is perfect in a constant concentration technique, $c_i^i = 0$.

$\underline{X_i}'$ is the vector containing the unknown flows to the zone i:

$$\underline{X_i}' = (Q_{0i}, Q_{1i}, \ldots, Q_{i-1,i}, 0, Q_{i+1,i}, \ldots, Q_{Ni}) \qquad (23)$$

and $\underline{\underline{C_i}}'$ a matrix having the element δc_{kj} in row k and column j:

$$\delta c_{kj} = \frac{c_j^k - c_i^k}{T_j} \qquad (24)$$

or, assuming H 5 and H 6:

$$\delta c_{kj} = c_j^k - c_i^k \qquad (25)$$

The matrix $\underline{\underline{C_i}}'$ has N rows and N+1 columns, but the column i is filled with zeroes. The vector $\underline{X_i}$ has also N+1 components, the component i, corresponding to Q_{ii} being zero. We can therefore contract the system (20) into:

$$\underline{Y_i} = \underline{\underline{C_i}}\ \underline{X_i} \qquad (26)$$

where $\underline{X_i}$ is the vector $\underline{X_i}'$ without the the component i, and $\underline{\underline{C_i}}$ the matrix $\underline{\underline{C_i}}'$ without the column i. The system (26) is then entirely determined if the matrix $\underline{\underline{C_i}}$ is not singular. It is solved in principle by:

$$\underline{X_i} = \underline{\underline{C_i}}^{-1}\ \underline{Y_i} \qquad (27)$$

which gives the N flows Q_{ji} from the zones j (j = 0 .. N) to the zone i. Solving (26) for the N zones, we get all the flows Q_{ji} where j = 0 .. N and i = 1 .. N. The N remaining flows Q_{i0} are computed using the N equations (16) or (18)

Note: This resolution method differs slightly from the one presented by Sinden [1] and Perera [2] but seems simpler if the constant concentration technique is used.

2.2 Discretisation

The time derivatives of the tracer concentration appear in the vector $\underline{Y_i}$, and the derivatives of the temperatures are used in equation (16). The equations (26) and (16) can be solved at each time interval, if we write either the backwards derivative:

$$\dot{c}_i^k(t) = [c_i^k(t) - c_i^k(t-\delta t)]/\delta t \qquad (28)$$

$$\dot{T}_i^k(t) = [T_i^k(t) - T_i^k(t-\delta t)]/\delta t \qquad (29)$$

or the forwards derivative:

$$\dot{c}_i^k(t) = [c_i^k(t+\delta t) - c_i^k(t)]/\delta t \qquad (30)$$

$$\dot{T}_i^k(t) = [T_i^k(t+\delta t) - T_i^k(t)]/\delta t \qquad (31)$$

For causality reasons, equations (28) and (29) may be better. A test is running with both possibilities on a set of measurements made at the LESO on a single cell. For this cell, using equation (19):

$$Q_{01} = \frac{T_0}{c_0 - c_1} \left[\frac{V_1}{T_1} \dot{c}_1 - \frac{q}{T} \right] \qquad (32)$$

where T is the temperature of the tracer when injected, T_1 the temperature of the air in the zone and T_0 the outdoor temperature.

In the reality, the measurements with the constant concentration techniques are done the following way:

- At a time t_0, a sample of the air of a defined zone is taken and analyzed and this operation lasts for the time δt_a.
- At time $t_0+\delta t_a$, the amount of tracer gas necessary to maintain a constant concentration is injected in the zone by opening a valve for a calculated time δt_{inj}.
- The start of the next sampling takes part some time after the end of this injection.

Hence, the sampling-analysis-injection cycle lasts a period of time which should be taken as a "natural" time step for the discretisation. Usually this time step is also the time interval between the records of measured data.

In this paper, all the variables (except the volumes of the zones) may depend on time but it is implicitely admitted that they are constant during the discretization time step. This assumption seems obvious, since we have no information of the variation of the concentrations between the measurements. Nevertheless, it shall be remembered that the time step may be large (e.g. 10 minutes) when compared to the time constants of the possible changes of the air- and tracer injection flows. Because of this, interpretation errors may occur, generally overestimating the air flow rates by several percent.

It is possible that a more refined interpretation method could be found, taking into account that the tracer injection rate is not constant during the measurement interval and that the tracer concentration may change during that interval. An indication of how this might be done can be found in the work Aittomäkki did for U-value measurements [6].

3 ERROR ANALYSIS

3.1 The problem

As a result of an experience, we have to solve a system of equations:

$$\underline{\underline{A}}\,\underline{x} = \underline{y} \tag{33}$$

where vector $\underline{y}$ and matrix $\underline{\underline{A}}$ both contain measured coefficients (y_j and a_{ij}) which are perturbated with errors represented respectively by a vector $\underline{\delta y}$ and a matrix $\underline{\underline{\delta A}}$. The question is: which is the resulting error $\underline{\delta x}$ on the vector $\underline{x}$?

Since the components x_i are functions of the coefficients a_{ij} and y_j, their variances can be computed approximately by:

$$s^2(x_i) = \sum_j \frac{\delta x_i}{\delta a_{ij}} s^2(a_{ij}) + \sum_j \frac{\delta x_i}{\delta y_j} s^2(y_j) \tag{34}$$

where $s^2(a_{ij})$ and $s^2(y_j)$ are the variances of the measured coefficients. If the errors in the variables a_{ij} and y_j are independant and symetrically distributed and if the function is not far from linear, this first order error analysis gives satisfactory results [9]. However the computation time required increases very strongly with the number of variables (that is the number of zones).

If the matrix $\underline{\underline{\delta A}}$ and the vector $\underline{\delta y}$ were exactly known for a given case, we could write:

$$(\underline{\underline{A}} + \underline{\underline{\delta A}})\,(\underline{x} + \underline{\delta x}) = \underline{y} + \underline{\delta y} \tag{35}$$

and, taking (33) into account, this could be solved:

$$\underline{\delta x} = (\underline{\underline{A}} + \underline{\underline{\delta A}})^{-1}\,(\underline{\underline{\delta A}}\,\underline{x} + \underline{\delta y}) \tag{36}$$

Equation (36) can however not be used to determine the absolute error or a statistical standard deviation. For that latter purpose, we can use (36) several times in a Monte-Carlo process, varying the components of $\underline{\underline{\delta A}}$ and $\underline{\delta y}$ at random but according to their probability density function. This provides several vectors $\underline{\delta x}$ from which an estimate of the probability density functions of the components can be calculated.

The vector $\underline{\delta x}$ contains a large number of values. It would be helpful to represent the error by a single figure. To obtain such a single figure, we need the following definitions, which can be found in the specific mathematical literature (e.g. [9]).

3.2 Vectorial and matrix norms

The norm $\|\underline{x}\|$ ***of a vector*** $\underline{x}$ is any operation of R^n in R satisfying:

$$\begin{aligned} &\|\underline{x}\| \geq 0 \text{ and } \|\underline{x}\| = 0 \text{ if and only if } \underline{x} = \underline{0} \\ &\|c\underline{x}\| = |c|\,\|\underline{x}\| \text{ for any } c \in R \\ &\|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\| \end{aligned} \tag{37}$$

For example, the euclidian norm (which corresponds to the standard deviation if the average $\langle x \rangle = 0$):

$$\|\underline{x}\|_2 = \sqrt{(\Sigma x_i^2)} \qquad (38)$$

fulfills the relations (37), but there are other norms, like $\|\underline{x}\|_1=\Sigma|x_i|$ or the infinity norm defined by $\|\underline{x}\|_\infty=\max|x_i|$.

The norm $\|\underline{\underline{A}}\|$ ***of a matrix*** $\underline{\underline{A}}$ is any application $N(\underline{\underline{A}})\to\|\underline{\underline{A}}\| \in R$ satisfying:

$$\begin{aligned} &\|\underline{\underline{A}}\| \geq 0 \text{ and } \|\underline{\underline{A}}\| = 0 \text{ if and only if } \underline{\underline{A}} = \underline{\underline{0}} \\ &\|c\underline{\underline{A}}\| = |c|\ \|\underline{\underline{A}}\| \text{ for any } c \in R \qquad (39) \\ &\|\underline{\underline{A}}+\underline{\underline{B}}\| \leq \|\underline{\underline{A}}\| + \|\underline{\underline{B}}\| \\ &\|\underline{\underline{A}}\cdot\underline{\underline{B}}\| \leq \|\underline{\underline{A}}\|\cdot\|\underline{\underline{B}}\| \end{aligned}$$

The matrix norm $\|\underline{\underline{A}}\|$ is *consistent* with the vectorial norm $\|\underline{x}\|$ if:

$$\|\underline{\underline{A}}\ \underline{x}\| \leq \|\underline{\underline{A}}\|\cdot\|\underline{x}\| \text{ for any } \underline{x} \qquad (40)$$

The matrix norm *subordinated* to the vectorial norm $\|\underline{x}\|$ is defined by:

$$\|\underline{\underline{A}}\| = \max(\|\underline{\underline{A}}\ \underline{x}\|/\|\underline{x}\|) \text{ for any } \underline{x} \neq \underline{0} \qquad (41)$$

The subordinated matrix norm is the smallest one which is consistent with $\|\underline{x}\|$. For example, the norm $\|\underline{\underline{A}}\|_2$, defined by:

$$\|\underline{\underline{A}}\|_2 = \sqrt{\mu_1} \qquad (42)$$

where μ_1 is the largest eigenvalue of $\underline{\underline{A}}^H\underline{\underline{A}}$ ($\underline{\underline{A}}^H$ = hermitic conjugate or the transpose of the complex conjugate matrix) is subordinated to the euclidian norm $\|\underline{x}\|_2$. This matrix norm is the ***spectral norm.***

Both Walker [7] and d'Ottavio [8] present an error analysis method similar to the following one, but they use other matrix and vectorial norms. Walker uses the vectorial infinity norm and the subordinated matrix norm which is the maximum row sum of the moduli of the elements. These norms lead to simpler computations but do not give a good image of the usual standard deviation. D'Ottavio et al. [8] use the Frobenius norm:

$$\|A\|_F = \sqrt{(\sum_{i,j}\sum(a_{ij})^2)} \qquad (43)$$

which is consistent with the euclidian vectorial norm but which is not subordinated to it, hence not the smallest. Therefor these authors have introduced a correcting factor adjusted to fit the results of this error analysis method to the results of the first order approximation method (equation 34).

It is proposed here to use the ***euclidian vectorial norm*** and the subordinated ***spectral matrix norm*** for the error analysis. The advantages are that the euclidian norm of the vector $\underline{\delta y}$ represents its total standard deviation and, using the subordinated matrix norm, we will obtain the smallest upper bound for $\underline{\delta x}$.

3.3 Error analysis

It is possible to give an upper limit to the norm of the resulting error $\|\underline{\delta x}\|$, using the following relations (from [9]) for the norms of the experimental errors $\|\underline{\delta y}\|$ and $\|\underline{\underline{\delta A}}\|$.

If only $\underline{y}$ is perturbed, we can write:

$$\underline{\underline{A}}\,(\underline{x} + \underline{\delta x}) = \underline{y} + \underline{\delta y} \tag{44}$$

but, taking (21) into account, we get:

$$\underline{\underline{A}}\,\underline{\delta x} = \underline{\delta y} \tag{45}$$

Then, for any pair of vectorial and matrix subordinated norms:

$$\|\underline{\delta x}\| \leq \|\underline{\underline{A}}^{-1}\,\underline{\delta y}\| \leq \|\underline{\underline{A}}^{-1}\|\cdot\|\underline{\delta y}\|$$

and

$$\frac{\|\underline{\delta x}\|}{\|\underline{x}\|} \leq \|\underline{\underline{A}}\|\cdot\|\underline{\underline{A}}^{-1}\|\cdot\frac{\|\underline{\delta y}\|}{\|\underline{y}\|} \tag{46}$$

The number:

$$\mathrm{cond}(\underline{\underline{A}}) = \|\underline{\underline{A}}\|\cdot\|\underline{\underline{A}}^{-1}\| \tag{47}$$

is of great importance here. It is the ***condition number*** of the matrix A related to the used norm. If $\|A\|_2$ is used, we get the smallest possible condition number. Its value is:

$$\mathrm{cond}_2(\underline{\underline{A}}) = \|\underline{\underline{A}}\|_2\cdot\|\underline{\underline{A}}^{-1}\|_2 = \sqrt{(\mu_1/\mu_n)} \tag{48}$$

where μ_1 and μ_n are respectively the largest and the smallest eigenvalues of $\underline{\underline{A}}^H\cdot\underline{\underline{A}}$. This condition number is the **spectral condition number**.

Matrix $\underline{\underline{A}}$ only is perturbed. In this case:

$$(\underline{\underline{A}} + \underline{\underline{\delta A}})(\underline{x} + \underline{\delta x}) = \underline{y} \quad \text{then} \quad (\underline{\underline{A}} + \underline{\underline{\delta A}})\underline{\delta x} = -\,\underline{\underline{\delta A}}\underline{x} \tag{49}$$

If $(\underline{\underline{A}} + \underline{\underline{\delta A}})$ is regular, hence if $\|\underline{\underline{A}}^{-1}\delta A\| \leq \|\underline{\underline{A}}^{-1}\|\cdot\|\underline{\underline{\delta A}}\| < 1$, then:

$$\frac{\|\underline{\delta x}\|}{\|\underline{x}\|} \leq \frac{\|\underline{\underline{A}}^{-1}\underline{\underline{\delta A}}\|}{1 - \|\underline{\underline{A}}^{-1}\underline{\underline{\delta A}}\|} \leq \frac{\|\underline{\underline{A}}\|\cdot\|\underline{\underline{A}}^{-1}\|\cdot\|\underline{\underline{\delta A}}\|/\|\underline{\underline{A}}\|}{1 - \|\underline{\underline{A}}\|\cdot\|\underline{\underline{A}}^{-1}\|\cdot\|\underline{\underline{\delta A}}\|/\|\underline{\underline{A}}\|} \tag{50}$$

$\underline{\underline{A}}$ and $\underline{b}$ are perturbed

In this case, assuming that $\|\underline{\underline{I}}\| = 1$ (which is true for $\|\underline{\underline{I}}\|_2$):

$$\frac{\|\underline{\delta x}\|}{\|\underline{x}\|} \leq \frac{\|\underline{\underline{A}}\|\cdot\|\underline{\underline{A}}^{-1}\|}{1 - \|\underline{\underline{\delta A}}\|\cdot\|\underline{\underline{A}}^{-1}\|}\cdot\left[\frac{\|\underline{\delta y}\|}{\|\underline{y}\|} + \frac{\|\underline{\underline{\delta A}}\|}{\|\underline{\underline{A}}\|}\right] \tag{51}$$

3.4 How to obtain a good condition

To minimize the condition number of the experimental matrix, some variable changes can be used on the initial problem in such a way that the coefficients a_{ij} of the matrix $\underline{\underline{A}}$ satisfy:

$$1/\beta \leq \max(|a_{ij}|) \leq 1 \qquad (52)$$

where β is the base of the floating point arithmetic system installed in the computer used to solve the system of equations. For that purpose, it is often enough to divide each row of the system by the largest coefficient, rounded to an integer power of β, to minimise the rounding errors.

Moreover, in the Gaussian resolution procedure, the equations are permutated in such a way that the largest (in absolute value) pivot is used.

3.5 Error analysis for two special cases

If e and ϵ are the ***constant relative errors*** on $\underline{\underline{A}}$ and $\underline{y}$:

$$\underline{\underline{\delta A}} = e\ \underline{\underline{A}} \text{ and } \underline{\delta y} = \epsilon\ \underline{y} \qquad (53)$$

and, from the definition of the norms:

$$\|\underline{\underline{\delta A}}\| = e\ \|\underline{\underline{A}}\| \text{ and } \|\underline{\delta y}\| = \epsilon\ y \qquad (54)$$

We have, for any definition of the norm satisfying $\|I\| = 1$:

$$\frac{\|\underline{\delta x}\|}{\|\underline{x}\|} \leq \frac{\text{cond}(\underline{\underline{A}})}{1 - e\ \text{cond}(\underline{\underline{A}})} \cdot (\epsilon + e) \qquad (55)$$

It should be remembered that $\underline{\underline{A}}+\underline{\underline{\delta A}}$ should be regular, which will occur if:

$$e \cdot \text{cond}(\underline{\underline{A}}) < 1 \qquad (56)$$

In the case of a ***constant absolute error***:

$$\underline{\underline{\delta A}} = e\ \underline{\underline{1}} \quad \text{and} \quad \underline{\delta y} = \delta y\ \underline{1} \qquad (57)$$

where $\underline{\underline{1}}$ and $\underline{1}$ are respectively a matrix of order N and a vector with N components where all the elements are unity.

It is easily seen that, if the euclidian and spectral norms are used:

$$\|\underline{\underline{1}}\|_2 = N \quad \text{and} \quad \|\underline{1}\|_2 = \sqrt{N} \qquad (58)$$

In fact, the eigenvalues of $\underline{\underline{1}}$ are N and 0^{N-1}, hence those of $\underline{\underline{1}}^H\underline{\underline{1}} = \underline{\underline{1}}^2$ are N^2 and 0^{N-1}. Then:

$$\|\underline{\underline{\delta A}}\|_2 = e\ N \text{ and } \|\underline{\delta y}\|_2 = \sqrt{N}\ \delta y \qquad (59)$$

and

$$\frac{\|\underline{\delta x}\|_2}{\|\underline{x}\|_2} \leq \frac{\text{cond}_2(\underline{\underline{A}})}{1 - e\ N \cdot \|\underline{\underline{A}}^{-1}\|_2} \cdot \left[\frac{\sqrt{N}\ \delta y}{\|\underline{y}\|_2} + \frac{Ne}{\|\underline{\underline{A}}\|_2} \right] \qquad (60)$$

We see that for these simple cases, which are the most common in practice, an upper bound for the relative error on the results can be easily calculated when the condition number of the matrix A is known. This matrix is $\underline{\underline{C}}_i$ in equation 26.

4 EXAMPLE

4.1 Case configuration

A constant concentration measurement on a very simple building with two rooms was simulated using a multizone infiltration computer code to provide the data used in this example. This way, the measurement errors are avoided. The input data were the following:

	Zone 0 (Outside)	Zone 1	Zone 2	
Temperature	0	20	20	C
Volume	-	50	100	m³
Flows from outside Q_{0i}	-	10	30	m³/h
Flows from zone 1 Q_{1i}	20.73	-	10	m³/h
Flows from zone 2 Q_{2i}	22.2	20	-	m³/h

The volume flows Q_{j0} to the outside result from the mass conservation of the air. After a few hours, the tracer concentrations stabilize and the results of the simulated measurements are:

	Zone 1	Zone 2	
Concentration of tracer 1	10	2.29	ppm
Concentration of tracer 2	6.46	10	ppm
Injection rate of tracer 1	$2.65\ 10^{-4}$	0	m³/h
Injection rate of tracer 2	0	$3.6\ 10^{-4}$	m³/h

4.1 Results for uniform temperature

Using equation (17) for the zone 1, we get:

$$\begin{bmatrix} -2.65\ 10^{-4} \\ 0 \end{bmatrix} = \begin{bmatrix} -10 & -7.71 \\ -6.46 & 3.54 \end{bmatrix} 10^{-6} \begin{bmatrix} Q_{01} \\ Q_{21} \end{bmatrix}$$

which gives the flows:

Q_{01} = 10.99 m³/h and Q_{21} = 20.06 m³/h

The same calculations for the zone 2 gives:

Q_{02} = 32.57 m³/h and Q_{12} = 9.68 m³/h

Finally, using equation 18 we get:

Q_{10} = 21.37 m³/h and Q_{20} = 22.19 m³/h

Comparing these results with the exact values given above, we see that the largest differences, of about 10 %, ocurr for Q_{01} and Q_{02}, which are the flows coming from the cold.

4.3 Results taking into account the temperatures

Using the new proposed equations (15) and (16), the following results are obtained:

Q_{01} = 10.21 m³/h Q_{21} = 20.06 m³/h
Q_{02} = 30.35 m³/h Q_{12} = 9.68 m³/h
Q_{10} = 21.37 m³/h Q_{20} = 22.19 m³/h

Since both zones have the same temperatures, only the flows coming from outside are changed and these results are more accurate that the results found under 4.2. The remaining differences (3% or less) are caused by rounding errors, since only three digits were kept for the simulated concentrations and tracer injection flow rates.

4.4 Error analysis

The matrices of equations 15 were reconditionned according section 3.4 before solving the system, in such a way that the largest elements (in absolute value) of each row were equal to 1. A uniform relative error was assumed, being 2% on the concentration matrix and 5% on the injection vector (that is $e = 0.02$ and $\epsilon = 0.05$) For this reconditionned system of equations and using two different norms, the following results were obtained, according equation 55 for $\|\underline{\delta Q}\|/\|Q\|$:

Norm used	Frobenius (eq. 43)		Spectral (eq. 42)	
Zone	1	2	1	2
Matrix norm $\|\underline{\underline{A}}\|$	1.666	1.487	1.426	1.057
and for the inverse $\|\underline{\underline{A}}^{-1}\|$	1.356	1.355	1.161	0.956
cond(A) = $\|\underline{\underline{A}}\|\cdot\|\underline{\underline{A}}^{-1}\|$	2.259	2.015	1.656	1.010
Relative error $\|\underline{\delta Q}\|/\|Q\|$	0.166	0.146	0.12	0.072

We see that the spectral norm gives a smaller upper bound for the relative error than the Frobenius norm, the difference going up to a factor of two for the second zone.

If the matrix were not reconditionned, we would get, for the zone 1 and with the spectral norm:

$$\|\delta Q\|/\|Q\| = 0.153$$

which is only slightly more than with the conditionned matrix in this case, since the system is already not too badly conditionned. For badly conditionned systems, this difference may be much larger.

CONCLUSIONS

In this paper, we have proposed to modify the equations of conservation of the tracer gases and of the air in such a way that measurements made with variable and non homogeneous temperatures can be interpreted properly.

Secondly, in order to obtain error figures which are not exaggeratedly large and which represent standard deviations, it is proposed to use the euclidian vectorial norm and the subordinated spectral matrix norm.

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