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The Evaluation of Contaminant Concentrations and Air Flows in a Multizone Model of a Building

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Improvements to tracer decay techniques for measuring flow rates in multi-cell buildings are proposed on the basis of a study of the governing equations. A detailed examination of the forward solution, in which tracer gas concentrations are predicted from known flow rates, has been carried out. This has revealed properties of the decay curves, which, if recognised, can assist in the extraction of flow rates from measured tracer concentrations. Proposals are made for tracer gas seeding strategy, and for computational procedures.

NOMENCLATURE

a_i concentration coefficient to ith eigenvector
f_{ij} $F_{ij}V_j^{-1}$
F_{ij} volumetric flow rate from zone i to zone j ($m^3\,s^{-1}$)
n number of zones
r_i $S_iV_i^{-1}$
S_i sum of flows into zone i ($m^3\,s^{-1}$)
V_i volume of zone i (m^3)
$c(t)$ concentration in zone i

$$\underline{c}(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{bmatrix}$$

$$\underline{x}_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{bmatrix} \quad i\text{th eigenvector}$$

Greek symbol
λ_i ith eigenvalue

1. INTRODUCTION

IN AIR movement studies it is convenient to represent a building as an assembly of interconnected zones, each zone being capable of exchanging air with any other zone. Usually, one zone is taken to represent external air, so that air movement between inside and outside may be represented as well as air movement within the building. If the interzonal flow rates are known, the multizone model may be used to compute the time evolution of the spread of an airborne contaminant. This may be particularly useful in certain types of buildings such as factories and hospitals. On the other hand, if flow rates are not known, the model can be used to obtain them from appropriate measurements of contaminant concentrations.

The differential equations governing contaminant distribution in a multizone model are well known, and have been given by Sinden [1] and Sandberg [2], both of whom make general remarks concerning their solution. However, in the particular case of the evaluation of flow rates from contaminant concentrations, as in the tracer decay method, it is of great advantage to understand the properties of the solution in considerable detail. Such an understanding is of value in, (i) determining the best initial distribution for the tracer gas, i.e. the most advantageous seeding strategy; (ii) avoiding poorly defined results due to ill-conditioning or linear dependency in the solution; (iii) maximising the information that can be obtained from a set of experimental data.

Perera and Walker [3] have considered some of these points, but their discussion is mostly confined to a particular building, and cannot, therefore, be easily generalized. Indeed, as will be shown later, the more general approach adopted here leads to conclusions which differ in some respects. The purpose of this paper is to examine the theory of the multizone air movement model in order to improve strategies for the derivation of interzonal air flows from tracer decay measurements. This has been done by examining in some detail the forward solution from known interzonal flow rates in order to identify pertinent features of the decay curves. These features are then used to sugest how both seeding strategy and the analysis of decay curves may be carried out to best advantage. The theory and discussion are restricted to the case of a single tracer gas.

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2. FUNDAMENTAL THEORY AND THE GENERATION OF DECAY CURVES FROM KNOWN FLOW RATES

The fundamental equations of a multi-zone air movement model have been stated by several authors, but are repeated here for completeness and consistency. Figure 1, due originally to ref. [1], and also given by ref. [2], illustrates the essentials of the model, in which a number of zones, $0, 1, \ldots, n$, are connected by one way passages through which air is flowing. The air in each zone is assumed to be fully mixed. Initially each zone contains a known concentration of tracer gas, and because we are restricting the problem to the tracer decay method, it is assumed that there is no generation of tracer gas in the system after time zero. Taking a volumetric balance on tracer gas in zone j, and balancing the total flow into and out of zone j gives

$$V_j \dot{c}_j(t) = \sum_{\substack{i=0 \\ i \neq j}}^{n} F_{ij} c_i(t) - c_j(t) S_j, \tag{1}$$

where S_j is the summation of the flows into or out of zone j, and is given by the conservation equation

$$S_j = \sum_{\substack{i=0 \\ i \neq j}}^{n} F_{ij} = \sum_{\substack{i=0 \\ i \neq j}}^{n} F_{ji}. \tag{2}$$

In matrix form, equation 1 becomes

$$V \underline{\dot{c}}(t) = F \underline{c}(t) \tag{3}$$

where

$$V = \begin{bmatrix} V_0 & 0 & \ldots & 0 \\ 0 & V_1 & \ldots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & V_n \end{bmatrix}$$

$$F = \begin{bmatrix} -S_0 & F_{10} & F_{20} & \ldots & F_{n0} \\ F_{01} & -S_1 & F_{21} & \ldots & F_{n1} \\ \vdots & & \ddots & & \\ F_{0n} & & & \ddots & -S_n \end{bmatrix}.$$

This is a system of of first order differential equations with the general solution

$$\underline{c}(t) = \sum_{k=0}^{n} a_k \underline{x}_k \, e^{\lambda_k t},$$

where $n+1$ values of λ_k and $\underline{x}_k$ are, respectively, the eigenvalues and eigenvectors of the equation

$$\lambda V \underline{x} = F \underline{x} \tag{4}$$

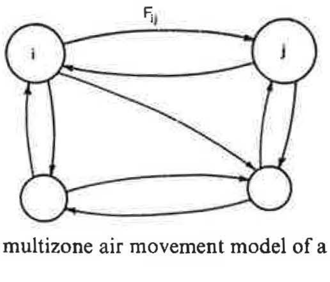

Fig. 1. The multizone air movement model of a building.

and the coefficients a_k are determined by the initial conditions. Sinden pointed out that (i) one eigenvalue, λ_0, is always zero, and that the corresponding eigenvector $\underline{x}_0$ is real and has equal components, (ii) all other eigenvalues and eigenvectors may be real or complex, (iii) complex values always occur in conjugate pairs, and (iv) all eigenvalues apart from λ_0 have negative real parts. Assigning infinite volume to zone 0 causes this zone to represent external air. The model then represents a building with n internal zones.

If tracer gas is not present in external air, the concentration in zone 0 is always zero. Hence $x_0(t)$ and the first row and first column of the vectors V and F may be deleted from Equation (2). The solution simplifies to

$$\underline{c}(t) = \sum_{k=1}^{n} a_k \underline{x}_k \, e^{\lambda_k t}. \tag{5}$$

The zero eigenvalue and its eigenvector no longer appear and the remaining λ_k and $\underline{x}_k$ are obtained from

$$\lambda V' \underline{x} = F' \underline{x}, \tag{6}$$

where

$$V' = \begin{bmatrix} V_1 & 0 & \ldots & 0 \\ 0 & V_2 & & \\ \vdots & & \ddots & V_n \end{bmatrix}$$

$$F' = \begin{bmatrix} -S_1 & F_{21} & \ldots & F_{n1} \\ F_{12} & -S_2 & & \\ \vdots & & \ddots & \\ F_{1n} & & & -S_n \end{bmatrix}.$$

Sinden's conditions (ii), (iii) and (iv) still apply. However, in the case of a two zone building ($n = 2$) only, Sinden showed that the two eigenvalues and their eigenvectors are always real. This indicates that for the two zone case, oscillatory solutions are impossible, whereas for all other cases ($n \geqslant 3$), oscillatory solutions exist for appropriate combinations of flow rates.

Solution of Equation (6) requires computation of the eigenvalues and eigenvectors of $E' = V'^{-1} F'$, which for $n > 3$ generally requires the use of numerical computation procedures. Such methods are certain to fail if E' has repeated eigenvalues and linearly dependent eigenvectors. That repeated eigenvalues can occur in quite normal situations can be demonstrated by some simple examples.

Example 1. *Repeated eigenvalues*, 2 *zone building*

For the general 2 zone building shown in Fig. 2, the eigenvalues can easily be shown to satisfy

$$\lambda = -\frac{1}{2}\left(\frac{S_1}{V_1} + \frac{S_2}{V_2}\right) \pm \frac{1}{2}\left[\left(\frac{S_1}{V_1} + \frac{S_2}{V_2}\right)^2 - 4\left(\frac{S_1 S_2}{V_1 V_2} - \frac{F_{12} F_{21}}{V_1 V_2}\right)\right]^{1/2} \tag{7}$$

from which it can be seen that repeated eigenvalues occur when

$$\left(\frac{S_1}{V_1} + \frac{S_2}{V_2}\right)^2 - 4\left(\frac{S_1 S_2}{V_1 V_2} - \frac{F_{12} F_{21}}{V_1 V_2}\right) = 0.$$

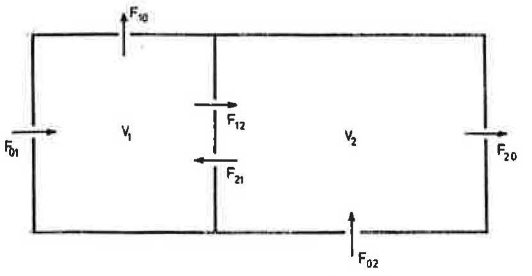

Fig. 2. The general 2 zone building.

Writing

$$r_1 = \frac{S_1}{V_1}, \quad r_2 = \frac{S_2}{V_2}, \quad f_{12} = \frac{F_{12}}{V_2}, \quad f_{21} = \frac{F_{21}}{V_1},$$

this becomes

$$(r_1 - r_2)^2 + 4 f_{12} f_{21} = 0.$$

Since all quantities are real non-negative numbers, this requires

$$(r_1 = r_2) \quad \text{and} \quad (f_{12} = 0 \quad \text{or} \quad f_{21} = 0).$$

The most obvious case which satisfies these conditions occurs when $V_1 = V_2$ and $F_{10} = F_{02} = F_{21} = 0$, which corresponds to a simple uni-directional flow of air through the building. The solution for $c_1(t)$ and $c_2(t)$ can be found directly as:

$$\left.\begin{aligned} c_1(t) &= a_1 e^{\lambda t} \\ c_2(t) &= (a_1 t + a_2) e^{\lambda t} \end{aligned}\right\} \tag{8}$$

where

$$\lambda = -\frac{S_1}{V_1} = -\frac{S_2}{V_2}.$$

More generally, if $V_1 \neq V_2$ and only $F_{21} = 0$, then the condition $r_1 = r_2$ will be met if

$$F_{12} = \frac{V_1}{V_2} F_{20} - F_{10}$$

and the solution for $c_1(t)$ and $c_2(t)$ becomes

$$\left.\begin{aligned} c_1(t) &= a_1 e^{\lambda t} \\ c_2(t) &= (a_1 f_{12} t + a_2) e^{\lambda t} \end{aligned}\right\} \tag{9}$$

Example 2. *Repeated eigenvalues,* 3 *zone building*

The eigenvalues will be the roots of a cubic equation

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0.$$

Again writing

$$r_i = \frac{S_i}{V_i} \quad \text{and} \quad f_{ij} = \frac{F_{ij}}{V_j},$$

it is a simple matter to show that

$$a = r_1 + r_2 + r_3$$

$$b = r_1 r_2 + r_2 r_3 + r_3 r_1 - f_{12} f_{21} - f_{23} f_{32} - f_{13} f_{31}$$

$$c = r_1 r_2 r_3 - r_1 f_{23} f_{32} - r_2 f_{13} f_{31} - r_3 f_{12} f_{21} - f_{12} f_{23} f_{31} - f_{13} f_{32} f_{21}$$

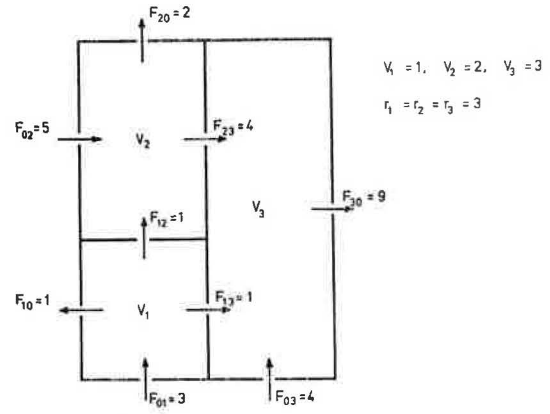

Fig. 3. A 3 zone building with repeated eigenvalues.

From elementary algebra, the cubic may be re-written as

$$x^3 + px + q = 0,$$

where

$$p = b - \frac{a^2}{3}, \quad q = c - \frac{ab}{3} + \frac{2a^3}{27} \quad \text{and} \quad \lambda = x - \frac{a}{3}.$$

The condition for repeated roots is that the discriminant is zero, i.e.

$$4p^3 + 27q^2 = 0.$$

Now, by substitution for a and b, p may be shown to be

$$p = -[f_{12} f_{21} + f_{23} f_{32} + f_{13} f_{31} + \tfrac{1}{6}\{(r_1 - r_2)^2 + (r_2 - r_3)^2 + (r_3 - r_1)^2\}].$$

Since all r_i and f_{ij} must be real and non negative, then $p \leqslant 0$, and real solutions to the discriminant exist. Thus combinations of r_i and f_{ij} exist which give repeated roots. In the particular case when all three roots are equal (and therefore real), then if this root is m, it may be shown that

$$b = \frac{a^2}{3}, \quad c = \frac{a^3}{27}, \quad p = q = 0, \quad \text{and} \quad m = -\frac{a}{3}.$$

Clearly $p = 0$ when $r_1 = r_2 = r_3$ and $f_{21} = f_{32} = f_{31} = 0$, and a typical building configuration where this might occur, with suggested values, is shown in Fig. 3. In this particular example the solution is

$$c_1(t) = \alpha e^{-3t}$$

$$c_2(t) = \beta e^{-3t} + \frac{\alpha}{2} t e^{-3t}$$

$$c_3(t) = \gamma e^{-3t} + \left(\frac{\alpha}{4} + \frac{4\beta}{3}\right) t e^{-3t} + \frac{\alpha}{3} t^2 e^{-3t}$$

where α, β, γ are determined by the initial conditions.

In general, the existence of repeated eigenvalues may be found by examining the Jordan canonical form of the matrix E'. Clearly the condition

$$r_1 = r_2 = \ldots = r_n \quad \text{with some} \quad f_{ij} = 0$$

is of particular significance, but whether or not this is a necessary condition has not been explored here. The possibility exists, therefore, that in general repeated eigenvalues will occur for a variety of conditions on r_i and f_{ij}. The physical significance of a repeated eigenvalue in the 2 and 3 zone examples given above is that one zone in the system does not receive a flow from any other zone (except zone 0 which is the outside), indicating that contaminant concentrations in this zone are unaffected by contaminant concentrations elsewhere in the building. It is possible that a repeated eigenvalue has the same physical significance in buildings of 4 or more zones.

3. PRINCIPAL PROPERTIES OF THE DECAY CURVES

The solution to the tracer decay problem, Equation (5), is in practice obtained in two stages. First, from the known interzone flow rates F_{ij} the eigenvalues and their associated eigenvectors are found. It is convenient to label the largest eigenvalue and its associated eigenvector λ_1 and $\underline{x}_1$ respectively. It should be noted that the eigenvalues and eigenvectors are completely determined by the set of F_{ij}. If one F_{ij} is altered, all λ_k and $\underline{x}_k$ are also altered. Secondly from the known initial contaminant concentrations, the coefficients a_k are calculated, each one being associated with one eigenvalue. Again, the a_k are determined by the set of initial conditions. Changing the initial conditions in any one zone will change all the a_k, but not the λ_k or $\underline{x}_k$.

The solution has a number of properties which, if recognised, can aid the interpretation of measured decay curves. The following points are of particular value.

(i) Whatever the initial conditions, a time will be approached when the largest eigenvalue dominates, after which all zones will decay at essentially the same rate, and the concentrations in the zones will be in the same ratio as the components of $\underline{x}_1$. Since by definition the contaminant concentrations in all zones are real positive numbers, it follows that λ_1 is a real negative number, and that a_1 and all components of $\underline{x}_1$ are real positive. The remaining eigenvalues, eigenvectors and coefficients may be real or complex. However, complex values always occur in conjugate pairs [Sinden's rule (iii)], and so if there are an even number of zones, there must always be at least one more real λ_k, $\underline{x}_k$ and a_k.

(ii) The time taken to reach the point at which the largest eigenvalue dominates, and a uniform decay is established in all zones depends on the initial distribution and the magnitude of the flow rates. If the initial distribution is such that the concentrations in the zones are in the same ratio as the corresponding components of $\underline{x}_1$, then uniform decay is established instantaneously, which implies that $a_1 = 1$ and $a_2 = a_3 = \ldots\ a_n = 0$. If the initial distribution is such that only the zone assoicated with the smallest component of the dominant eigenvector is contaminated, the time to reach uniform decay rate is maximised.

(iii) Starting from a non-uniform initial distribution, especially where only one zone is contaminated, it is likely that the other zones will show an increase in contaminant concentration in the early part of the process. However, conservation considerations require that the zone with the highest concentration at any given time must be decaying. Thus the concentration in any zone where it is rising must reach a peak not later than the time at which it equals the concentration in the zone which until then had shown the highest value. In particular, if the concentration versus time curves for any two zones in a system is such that the decaying concentration in one of them (say zone 1) intersects the rising concentration in the other (say zone 2) when the concentration in 2 is at its peak, then immediately it may be possible to infer that the only non-zero flow into 2 is from 1. This occurs frequently in practical situations, and it may be proved by writing the flow equation for zone 2 (from equation 1) in the form

$$c_2(t) = \frac{\Sigma F_{i2}c_i(t) - V_2\dot{c}_2(t)}{S_2},$$

or more simply, since at the point in question, $\dot{c}_2(t) = 0$,

$$c_2(t) = f_{02}c_0(t) + f_{12}c_1(t) + f_{22}c_2(t) + \ldots f_{n2}c_n(t)$$

The coefficients f_{ij} must be such that

$$f_{i2} = \frac{F_{i2}}{S_2}, \ \sum_{i=0}^{n} f_{i2} = 1, \quad 0 \leqslant f_{i2} \leqslant 1 \quad \text{and } f_{22} \equiv 0.$$

Now if at the point of intersection $c_1(t) > c_i(t)$ for all other i (except $i = 2$), the equality $c_2(t) = c_1(t)$ can only be satisfied if all f_{i2} are zero except f_{12}, which proves that F_{12} is the only non-zero flow *into* zone 2. The condition that $c_1(t)$ is greater than all other $c_i(t)$ may be relaxed for those $c_i(t)$ where it is already known from other considerations (e.g. building geometry) that the corresponding f_{i2} is zero. Thus, if $c_1(t) = c_2(t)$ when $\dot{c}_2(t) = 0$, it follows that F_{12} is the only flow into zone 2 if $c_1(t)$ is the maximum in the subset of zones which have a possible connection to zone 2.

(iv) It is not possible to associate any eigenvalue with an individual zone, neither is it possible to associate any flow rate with an eigenvalue. This is an obvious consequence of the fact than an alteration in only one of the F_{ij} will alter all the λ_k. In particular, the dominant eigenvalue which governs the later stages of decay does not relate to the overall fresh air infiltration rate of the whole building. However when internal interzonal flow rates become very large relative to the infiltration flow rates, the building will approximate to a fully mixed single zone, and in these circumstances λ_1 may become a good measure of the overall infiltration rate.

4. AN ILLUSTRATIVE EXAMPLE

The properties discussed in Section 3 can be illustrated by a hypothetical but nevertheless realistic example. Consider a four zone building, as shown in Fig. 4, in which the significant infiltration openings are

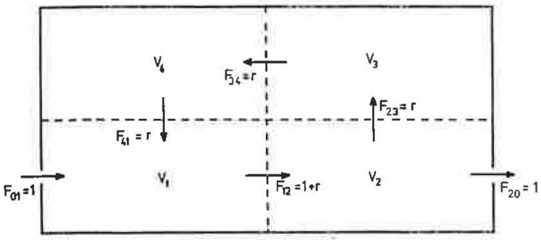

Fig. 4. A 4 zone building example.

concentrated in zones 1 and 2. Wind pressure will create a flow into zone 1, a flow from zone 1 to zone 2, and an outflow from zone 2. Superimposed on this there may be a circulation through zones 3 and 4, generated perhaps by entrainment or by internal stirring mechanisms. This example is a simplified model of a common natural ventilation situation; Fig. 4 may be visualized as a plan view, in which zones 3 and 4 are rooms with no direct link to the outside, or, as in an industrial building, the figure could represent a vertical section, in which zones 3 and 4 are the roof space.

Taking the total volume of the building as 3600 m^3, then with $F_{01} = F_{20} = 1$ m^3 s^{-1}, the fresh air infiltration rate is 3600 m^3 h^{-1} or 1 air change per hour. For values of the circulation flow rate, r greater than zero, zonal concentrations may be calculated from Equation (5), using values of λ_k and $\underline{x}_k$ found from Equation (6). This has been done for $r = 0.25$, $r = 1$ and $r = 4$ m^3 s^{-1}, giving a range of values from well below to well above the infiltration flow rate. The resulting eigenvalues and their eigenvectors are shown in Table 1. When $r = 0$, zones 3 and 4 are isolated from zones 1 and 2, and become stagnant areas, leaving a uni-directional flow through zones 1 and 2, exactly as in example 1 in Section 2. In this case $c_3(t) = c_4(t) = 0$, and the solutions for $c_1(t)$ and $c_2(t)$ are given by Equations 8, with

$$\lambda = -\frac{S_1}{V_1} = -\frac{S_2}{V_2} = -\frac{1}{900}.$$

The coefficients a_k have also been computed for a range of cases, illustrating the effect of different initial conditions. The values of a_k are shown in Table 2, in which the notation 1, 0, 0, 0 indicates that initially zone 1 was seeded with contaminant, whereas zones 2, 3 and 4 were contaminant free. The time evolution of the contaminant concentrations have also been calculated for these cases, and the results are plotted in Figs 5–11. Inspection of Tables 1 and 2 show, as explained in paragraph 3 (i), that λ_1 is always real negative, and that a_1 and all components of $\underline{x}_1$ are real positive. Where complex values arise, they are in conjugate pairs, and because there are an even number of zones, there are always in total at least two real sets of λ_k, $\underline{x}_k$ and a_k.

The effect of initial distribution on the time taken to reach a uniform decay is shown in Figs 5–9. The components of the dominant eigenvector are, from Table 1, in the order $x_{11} < x_{12} < x_{13} < x_{14}$. Therefore, from 3 (ii), the time to reach uniform decay from the seeding of a single zone should be greatest if zone 1 is seeded, becoming less in order as zone 2 or 3 or 4 is the zone seeded. When all zones are equally seeded the time to uniformity should be least. Inspection of the graphs tends to confirm this; the time at which all four zone concentrations settle into their final relative order [i.e. $c_4(t) > c_3(t) > c_2(t) > c_1(t)$] is a useful indication. However, a better criterion is to compare the ratios of the concentrations at a suitable point in time (say 2 time constants) with the components of $\underline{x}_1$. The closer these ratios are to $\underline{x}_1$, the closer is the decay process to uniformity. The comparison is shown in Table 3, which confirms the expectation. In practice, it can be seen that if, initially, all zones are uniformly contaminated, the time taken to reach uniform decay will always be short, whereas if, initially, any one zone only is contaminated, the time to uniform decay will be long. The effect of flow rate on the speed with which uniformity is established is shown in Figs 10 and 11, which show the effect of increasing r to 1 and then 4 for the 1, 0, 0, 0 case.

The effect described in 3 (iii) is most clearly shown in Figs 5, 10 and 11, in which seeding is in zone 1. In each

Table 1. Eigenvalues and eigenvectors for 4 zone building

	$i = 1$		$i = 2$		$i = 3$		$i = 4$	
				Case 1, $r = 4$				
λ_i	-2.68×10^{-4}	0	-5.00×10^{-3}	4.67×10^{-3}	-5.00×10^{-3}	-4.67×10^{-3}	-9.73×10^{-3}	0
x_{i1}	0.840	0	0.1	0.839	0.1	−0.839	−0.752	0
x_{i2}	0.882	0	1	0	1	0	1	0
x_{i3}	0.939	0	−0.111	−0.939	−0.111	0.939	−0.840	0
x_{i4}	1	0	−0.869	0.210	−0.869	−0.210	0.705	0
				Case 2, $r = 1$				
λ_i	-2.33×10^{-4}	0	-1.67×10^{-3}	1.20×10^{-3}	-1.67×10^{-3}	-1.20×10^{-3}	-3.10×10^{-3}	0
x_{i1}	0.558	0	0.250	0.539	0.250	−0.539	−0.395	0
x_{i2}	0.623	0	1	0	1	0	1	0
x_{i3}	0.790	0	−0.354	−0.763	−0.354	0.763	−0.558	0
x_{i4}	1	0	−0.458	0.539	−0.458	−0.539	0.311	0
				Case 3, $r = 0.25$				
λ_i	-1.40×10^{-4}	0	-4.64×10^{-4}	0	-1.20×10^{-3}	0	-1.53×10^{-3}	0
x_{i1}	0.223	0	0.300	0	0.134	0	−0.0995	0
x_{i2}	0.247	0	0.452	0	1	0	1	0
x_{i3}	0.498	0	−0.672	0	−0.300	0	−0.223	0
x_{i4}	1	0	1	0	0.0903	0	0.0495	0

Table 2. Coefficients for 4 zone building

	Initial distribution zone				Real and imaginary components of coefficients							
r	1	2	3	4	a_1		a_2		a_3		a_4	
0.25	1	0	0	0	2.24	0	−4.21	0	4.66	0	−4.53	0
0.25	0	1	0	0	0.202	0	−0.280	0	0.662	0	0.450	0
0.25	0	0	1	0	0.909	0	−0.931	0	0.421	0	−0.224	0
0.25	0	0	0	1	0.450	0	0.627	0	−1.40	0	1.01	0
0.25	1	1	1	1	1.785	0	−1.01	0	4.30	0	−3.39	0
1	1	0	0	0	0.274	0	0.354	−0.299	0.354	0.299	−0.879	0
4	1	0	0	0	0.263	0	0.070	−0.289	0.070	0.289	−0.372	0

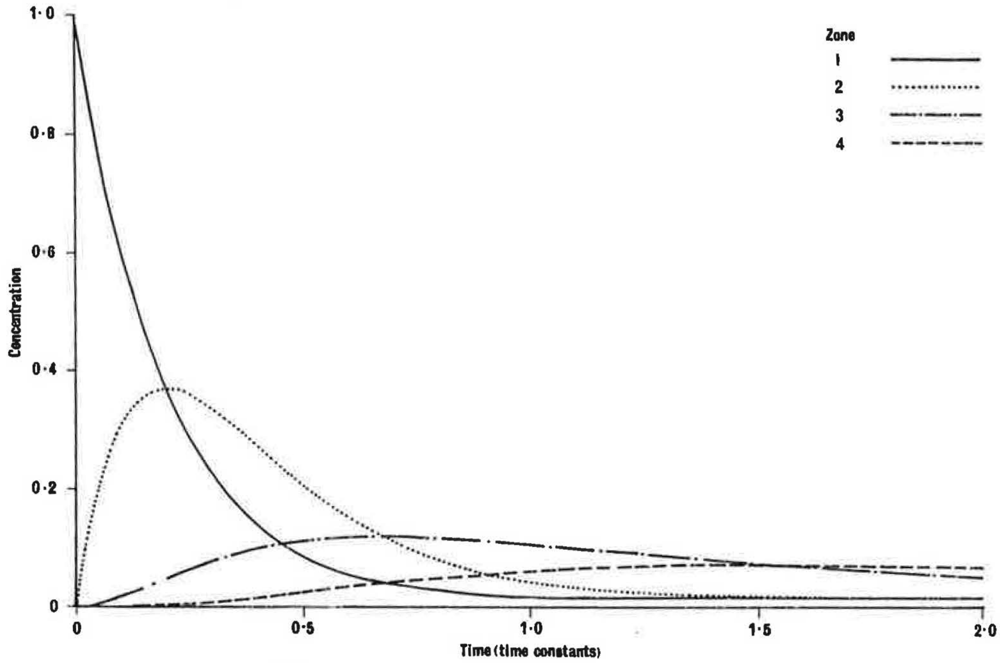

Fig. 5. Tracer decay, 4 zone building, $r = 0.25$, zone 1 seeded.

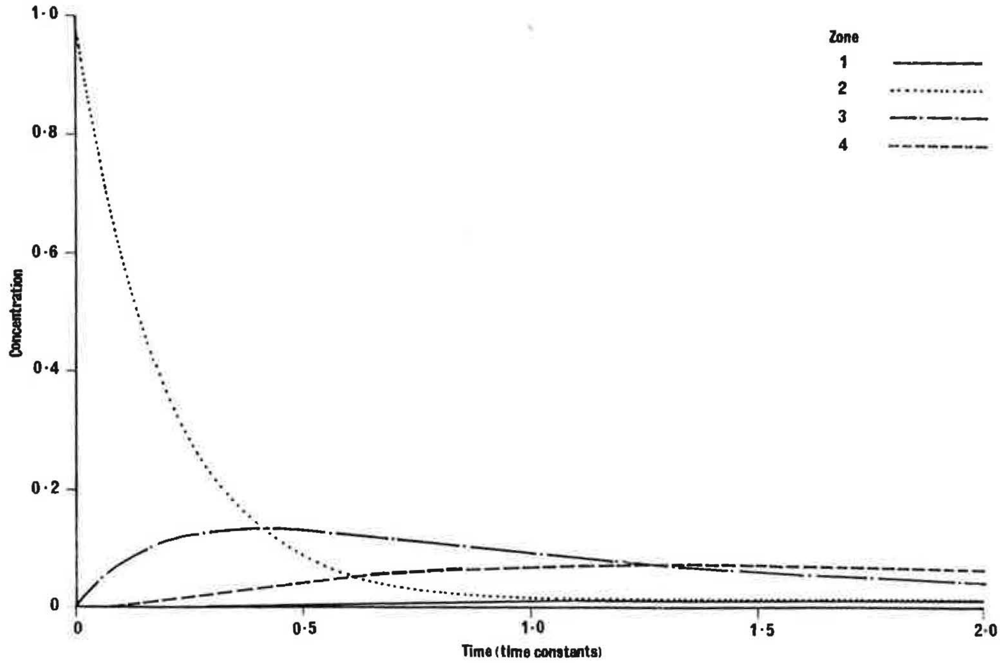

Fig. 6. Tracer decay, 4 zone building, $r = 0.25$, zone 2 seeded.

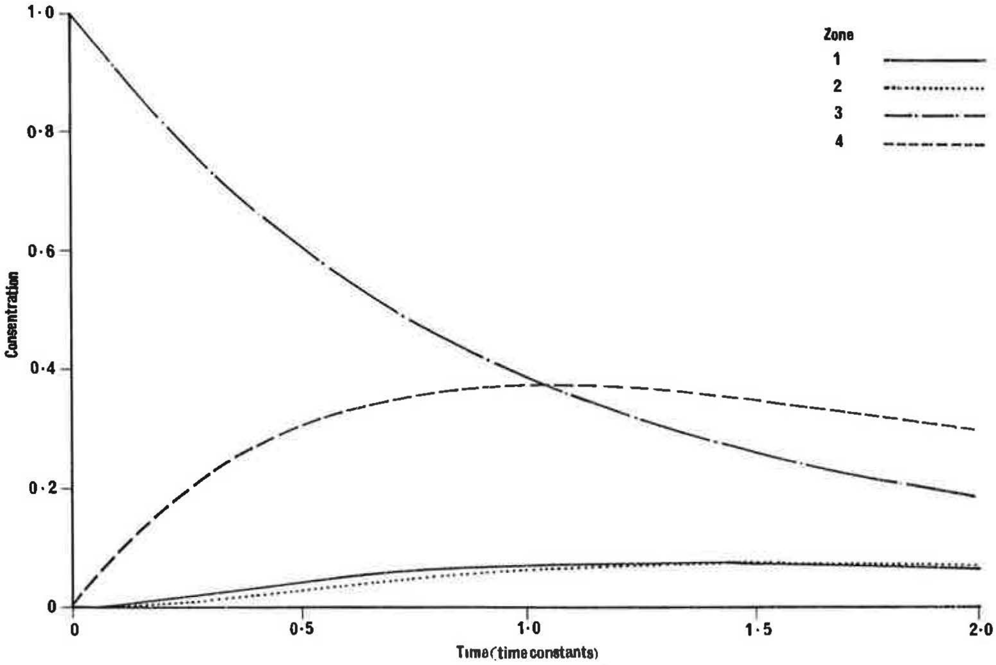

Fig. 7. Tracer decay, 4 zone building, $r = 0.25$, zone 3 seeded.

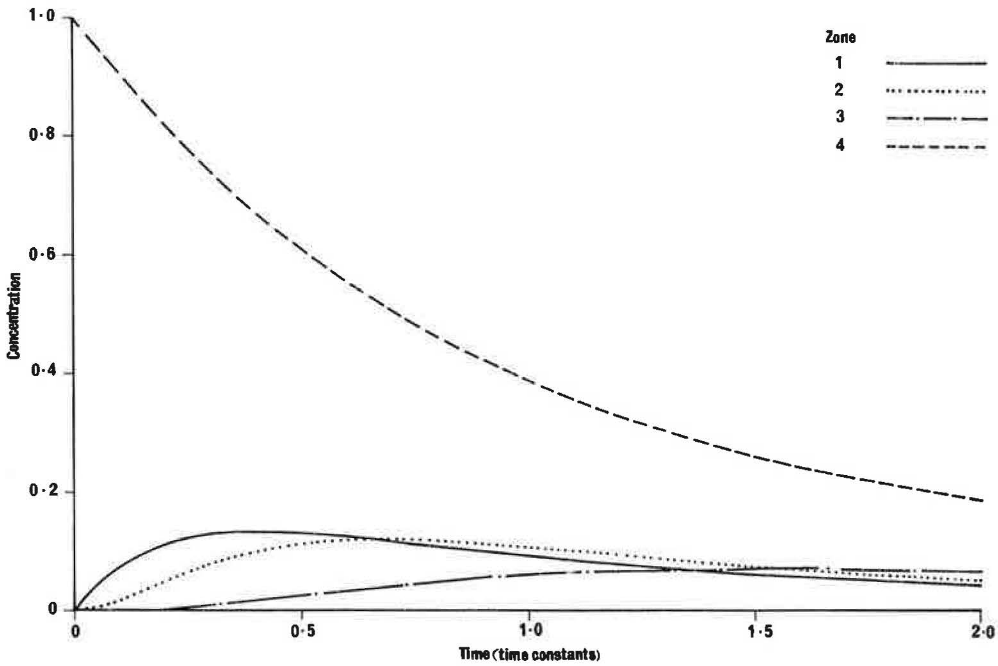

Fig. 8. Tracer decay, 4 zone building, $r = 0.25$, zone 4 seeded.

Table 3. Difference between concentration ratios and components of the dominant eigenvector at 2 time constants

		Seeded zone							
		1		2		3		4	
Zone i	x_{1i}	c_i	$\|c_i-x_{1i}\|$	c_i	$\|c_i-x_{1i}\|$	c_i	$\|c_i-x_{1i}\|$	c_i	$\|c_i-x_{1i}\|$
1	0.223	0.207	0.016	0.210	0.013	0.214	0.009	0.231	0.008
2	0.247	0.213	0.034	0.218	0.029	0.225	0.022	0.270	0.023
3	0.498	0.753	0.255	0.677	0.179	0.626	0.128	0.359	0.139
4	1	1	0	1	0	1	0	1	0
$\Sigma\|c_i-x_{1i}\|$			0.305		0.221		0.159		0.170

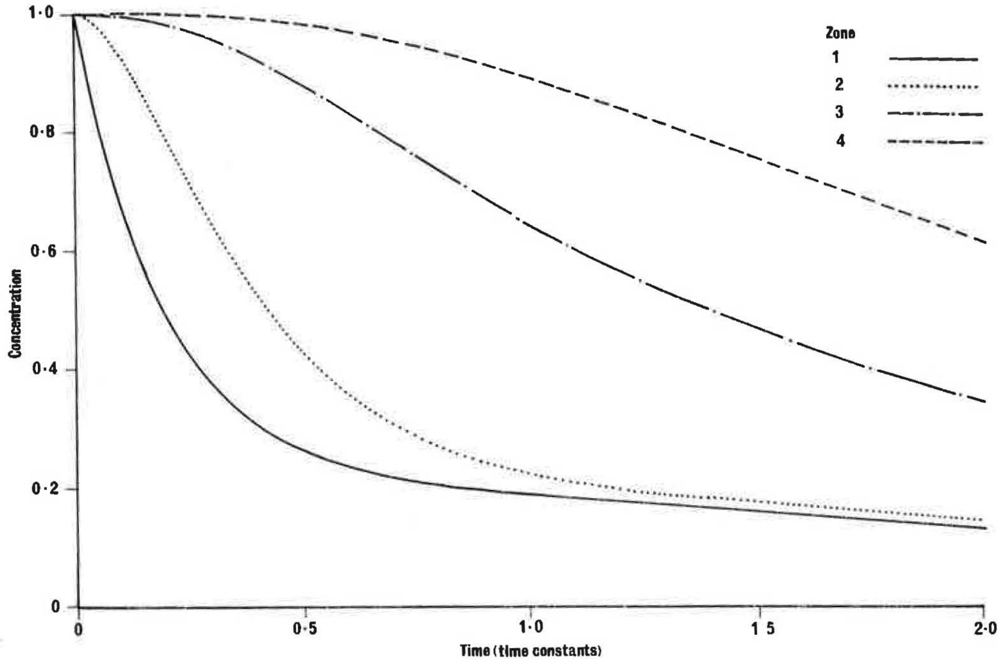

Fig. 9. Tracer decay, 4 zone building, $r = 0.25$, all zones uniformly seeded.

of these, the decaying concentration in zone 1 intersects the concentration in zone 2 when the latter is at its peak, indicating that all F_{i2} are zero except for F_{12}. Similarly, zone 2 intersects zone 3, and then zone 3 intersects zone 4, indicating that all F_{i3} and F_{i4} are zero except for F_{23} and F_{34}. Note also that seeding other zones gives less information. In Fig. 8, the intersection of $c_1(t)$ with $c_2(t)$ occurs when $\dot{c}_2(t)$ is zero, but it is necessary to assume that the diagonal flow F_{42} is zero in order to infer that F_{12} is the only flow into zone 2. Also in Fig. 8, because geometry does not preclude that $F_{43} = 0$, the fact that $c_2(t)$ intersects $c_3(t)$ when $\dot{c}_3(t)$ is zero does not necessarily require F_{23} to be the only flow into zone 3.

The relationship between the dominant eigenvalue, λ_1, and fresh air infiltration rate, as discussed in paragraph 3 (iv), can be found by multiplying λ_1 by 3600 to obtain air changes per hour. Table 4 shows how λ_1, expressed in units of h^{-1}, varies with r. Even with $r = 4$, the infiltration measured from the long term slope of the decay curves would be 4% below the true value of 1 air change

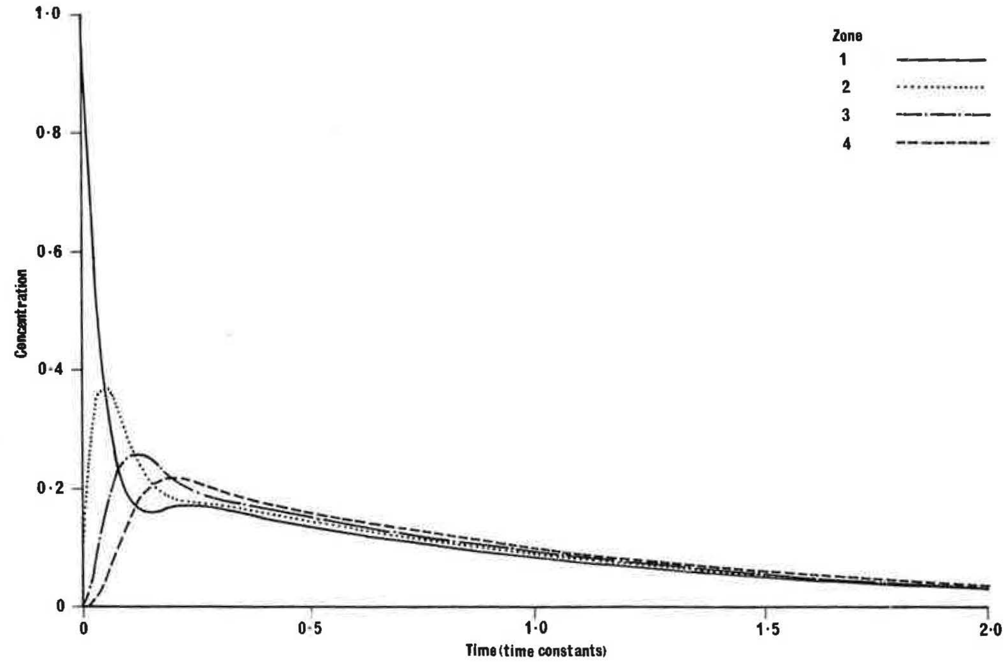

Fig. 10. Tracer decay, 4 zone building, $r = 1$, zone 1 seeded.

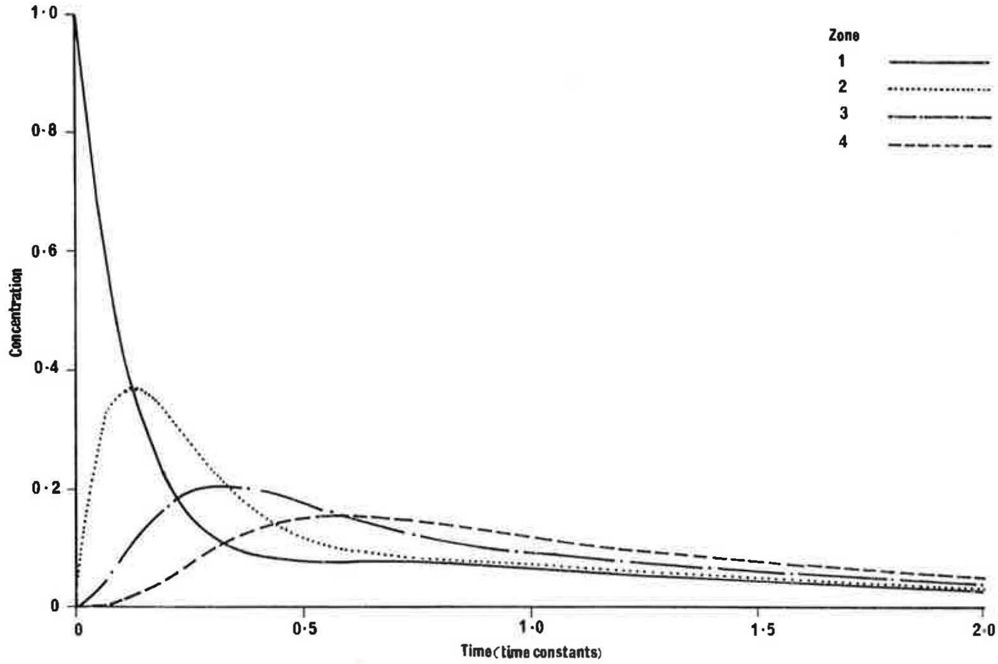

Fig. 11. Tracer decay, 4 zone building, $r = 4$, zone 1 seeded.

per hour, whereas at $r = 0$, it would be only 25% of the true value. Whilst it is obvious from this example that, for $r = 0$, the value of λ would give the correct fresh air infiltration rate if it were associated with the volume of zone 1 only, this cannot be done from the decay curves alone, without first establishing that F_{21}, F_{31} and F_{41} are all zero.

Paragraph 3(iv) above, and the results of this example throw some light on remarks made by Perera and Walker [3] concerning the association of eigenvalues with particular flow rates. Using computations from their own five-zone building, they concluded that "eigenvalues do not necessarily equate either to the fresh air infiltration or to the total air change rate of a zone". This is in agreement with the work presented here. However, they also state that the dominant eigenvalue must represent air movement between the building and the outside, and that the dominant eigenvalue is relatively unaffected by changes in interzone flows. This must be incorrect as it is clearly contrary to both the theory and example considered here.

5. THE EXTRACTION OF FLOW RATES FROM TRACER DECAY MEASUREMENTS

The problem encountered in practice is the opposite of that considered so far i.e. knowledge is required about

Table 4. Apparent fresh air infiltration rate, 4 zone building

r	λ_1	Air change rate (h^{-1})
4	-2.68×10^{-4}	0.96
1	-2.33×10^{-4}	0.84
0.25	-1.40×10^{-4}	0.50
0	-1.11×10^{-3}	0.25

air movement patterns within buildings and across their external fabric from tracer gas decay observations. In a system of n zones each connecting with the outside there are n^2+n flows, F_{ij}. The necessary n^2+n equations may be formed from the n flow conservation equations (2) and measurement of the concentrations $c_1(t), \ldots, c_n(t)$ and their derivatives $\dot{c}_1(t), \ldots, \dot{c}_n(t)$ on n occasions since each set of measurements yields n tracer gas conservation equations (1).

If the c_i data set is error free and gives perfect tracer decay curves, there is no difficulty in solving for the F_{ij} providing the $c_i(t)$ and $\dot{c}_i(t)$ can be obtained at n sufficiently different points in time to yield the necessary n^2 equations. There are however certain precautions that must be taken in order to avoid an inaccurate solution.

(i) As explained in Section 4, irrespective of initial conditions, zonal concentrations tend to relative magnitudes equivalent to the components of $\underline{x}_1$. If more than one set of $c_i(t)$ and $\dot{c}_i(t)$ are taken as this condition is approached, an ill conditioned set of equations and hence an inaccurate solution will result. It follows that it is necessary to ensure that adequate time is available for collecting well conditioned data, i.e. before the equilibrium concentrations are approached. This is most easily achieved by strategic seeding of the zones, especially, as explained in Section 4, by seeding the zone with lowest equilibrium concentration. Although this zone may often be identified intuitively, it is most easily obtained from the results of a preliminary set of measurements based on an arbitrary seeding pattern which is allowed to run until equilibrium concentration ratios are apparent.

(ii) Care must be taken when seeding buildings of symmetrical layout. In the case of the building shown in Fig. 12, if zone 2 alone was initially seeded the

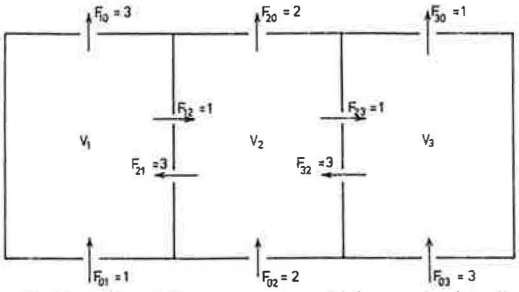

Fig. 12. Example of flow symmetry which may lead to linear dependence.

concentration in zone 1 would, for all time, be greater than that of zone 3 in the ratio of 3 to 1. As a consequence there would be linear dependence between the conservation equations and hence no unique solution. In the case of a real building, experimental scatter would mask the problem causing the linear dependence to give way to ill conditioning and an inaccurate solution.

This problem may be quite easily overcome by ensuring that the symmetry does not exist. In the above example, seeding zone 1, zone 3 or any combination of more than one zone instead of zone 2 alone, would be satisfactory.

(iii) Situations where there may be repeated eigenvalues, as in examples 1 and 2 in Section 2, may create difficulties if the wrong zone is seeded. In those particular examples, it is obvious that it is necessary to seed zone 1, otherwise not all the decay curves will be present. In the case of a real building where there is no pre-knowledge of the flow pattern, incorrect seeding may fail to reveal an important flow. In example 2, if zone 2 was seeded, there would be no measured tracer in zone 1. Nevertheless, a valid set of measurements would be obtained, the solution of which would lump the F_{12} flow in with F_{02}.

The solutions to examples 1 and 2 show that the solutions for the concentrations are not wholly independent, as $c_2(t)$ and $c_3(t)$ can be expressed in terms of $c_1(t)$. However, this dependency varies with time, and so it is still possible to obtain the necessary n^2 equations from the tracer conservation equation. In the 2 zone case, if t_1 and t_2 are the two times at which $c_1(t)$, $\dot{c}_1(t)$, $c_2(t)$ and $\dot{c}_2(t)$ are measured, then it may be shown by substituting equations 8 into equation 1, that the determinant of the coefficient matrix for the F_{ij} is exactly $(t_1-t_2)^2$. As this is only zero when $t_1 = t_2$ a unique solution should always be possible.

In practice, an experimental data set is subject to scatter from a variety of sources, and in many cases experimental technique provides measured values at discrete points in time only. As the direct solution technique relies on measurement of gradient as well as magnitude, the resulting errors in the F_{ij} may be substantial. Walker [4] has attempted an analysis of errors by considering suitable norms of the appropriate matrices. This is useful for examining the theoretical upper bounds of the errors, but does not help to quantify the probable error arising from a particular data set. Nevertheless, from the analysis presented in this paper, it is possible to deduce in a qualitative sense a series of measures which should reduce the effect of errors in the data set and improve the quality of the computed F_{ij}. Seeding strategy has already been discussed, and there are some other possibilities.

5.1. *Noise on the data*

If the data set is large enough, it is possible to use Sinden's suggestion, and integrate Equation (11) over different time intervals. Penman and Rashid (5) have used this method. The disadvantage is that decisions must be made concerning the length of the time intervals and their positions in the data set. There is no simple criterion which will resolve the conflicting requirements of long time intervals (to maximise noise suppression) and sufficiently different time intervals to avoid ill conditioning. This difficulty is avoided if a straightforward smoothing technique is adopted. For example if the data points are equally spaced in time, the method of fourth differences [6] may be applied. This gives smoothed values of both $c_i(t)$ and $\dot{c}_i(t)$ according to simple algebraic formulae. Where the smoothing is carried out over, say, five adjacent data points, we have:

$$C_i(t) = c_i(t) - 3/35[c_i(t-2s) - 4c_i(t-s) + 6c_i(t) - 4c_i(t+s) + c_i(t+2s)]$$

$$\dot{C}_i(t) = \frac{-2c_i(t-2s) - c_i(t-s) + c_i(t+s) + 2c_i(t+2s)}{10s}$$

where $C_i(t)$ and $\dot{C}_i(t)$ are the smoothed values of $c_i(t)$ and $\dot{c}_i(t)$, and s is the time interval between successive points. This method has been used in ref. [7].

5.2. *Reducing the number of unknowns*

It is rare in practical situations for all possible flows, F_{ij}, to exist, and therefore some of the F_{ij} can be set to zero from geometrical considerations alone. Of the remaining F_{ij}, some may also be set to zero if it is observed that any of the decay curves intersect as in paragraph 3 (iii). Furthermore, when a single zone is seeded, additional information can be obtained from an examination of the decay curves in the neighbourhood of the origin. If only one zone is seeded, say zone i, then at $t = 0$, this will be the only zone to contain tracer gas, and Equation (1) reduces to

$$V_i\dot{c}_i(0) = -c_i(0)S_i.$$

Thus

$$S_i = -V_i\frac{\dot{c}_i(0)}{c_i(0)} \tag{10}$$

and the limiting value as $t \to 0$ can conveniently be found by plotting the ratio $\dot{c}_i(t)/c_i(t)$ against time. Also, for any zone, j, which has a flow connection to zone i, equation 1 at time $t = 0$ can be reduced to

$$V_j\dot{c}_j(\theta) = F_{ij}c_i(0)$$

and so

$$F_{ij} = V_j\frac{\dot{c}_j(0)}{c_i(0)}.$$

However, this likely to over-estimate F_{ij} because time lags in the flow paths make it difficult to synchronise the measurements of $\dot{c}_j(0)$ and $c_i(0)$. If $c_i(0)$ is taken as the value of $c_i(t)$ at the time (usually slightly after $t = 0$) that the measured gradient in zone j is a maximum, then $c_i(0)$ is probably too low, and it is safer to write

$$F_{ij} \leqslant \frac{V_j \dot{c}_j 0}{c_i(0)}. \tag{11}$$

Again, it is convenient to plot the ratio of $\dot{c}_j(t)/c_i(t)$ against time.

5.3. *Application of least squares methods to measured data sets*

Once the data has been smoothed, and the number of unknown F_{ij}'s has been reduced to a minimum, a solution of Equations 1 and 2 may be attempted. A least squares technique is appropriate, but in multi-zone buildings it is often found that the optimum solution contains negative values for some of the F_{ij} [5, 7]. Clearly negative values of F_{ij} are physically impossible, and so it is necessary to apply the constraint $F_{ij} \geqslant 0$. Penman and Rashid achieved this by using the constrained least squares method described in ref. [8], and found that they were then able to obtain satisfactory results. Lawrance has done likewise, but has taken the idea further by including the extra information obtained from Equations 10 and 11 as additional constraints. The least squares solution could possibly be further refined if the equations obtained from the data set were weighted according to their position in the time series, in order to compensate for the fact that, as time progresses, the equations approach linear dependence. This, however, would require a criterion for evaluating appropriate weighting factors.

5.4. *Time delays*

In large buildings, zones may be sufficiently far apart for there to be a significant time lag between zones. This not only has the effect of displacing the decay curves for the zones by different amounts with respect to time, but can also affect the pattern of the decay process. This effect has been examined for a simple two zone building [9]. For the two zone case, it was found that the introduction of time lags creates an oscillation on the decay curves. Similar effects could occur in multizone cases, and may therefore need to be included in the analysis.

6. CONCLUSION

Solutions to the equations for the distribution of a contaminant in a multizone air movement model have been examined in detail, with a view to improving the evaluation of interzone flow rates from measured decay curves. The principal conclusions are;

(i) the most advantageous seeding strategy is to seed a single zone. Preferably this should be the zone in which the tracer concentration will as time progresses fall below the concentration in any other zone.

(ii) The number of unknown F_{ij}'s can be reduced not only from considerations of building geometry, but also if the decay curves exhibit certain features.

(iii) The values of all non-zero F_{ij}'s can be constrained to be greater than zero, and some can be constrained to other values or to an upper limit from close examination of the decay curves.

(iv) It is not possible to determine the overall fresh air infilitration of a building by measuring the dominant eigenvalue from the final uniform decay rate. The fresh air infiltration rate can only be found by first solving for the F_{ij}, and then summing to find

$$S_0 = \sum_{i=1}^{n} F_{i0} = \sum_{i=1}^{n} F_{0i}.$$

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