Technical Note

Summary Multi-zone models of air movement in buildings usually assume that there is no time lag in the flows between zones. Nevertheless, such time lags could have a significant effect on the pattern of contaminant distribution throughout a building. This note shows how such effects can be evaluated by examining theoretically a two-zone model with time lags. The results have implications for the interpretation of tracer decay measurements.

Time lags in a two-zone air movement model

J R WATERS BSc MPhil PhD MCIBSE
Department of Civil Engineering and Building, Coventry Lanchester Polytechnic, Priory Street, Coventry CV1 5FB, UK

1 Introduction

In multi-zone air movement models it is assumed that a building can be represented by an assembly of zones with the air in each zone perfectly mixed. The flows between zones are usually assumed to occur without time lags, that is, a particle of air in the flow from zone 1 to zone 2 will, on leaving zone 1, reappear instantaneously in zone 2. This assumption is implicit in the multi-zone air movement model of Sinden[1], and is perpetuated by authors such as Sandberg[2], Perera[3] and Waters and Simons[4] who have used Sinden's model. In practice, zones may be remote enough from one another for time lag effects to become important, especially if the principal connecting path is a corridor or duct. The effect of time lags will be most significant in a transient process, such as the spread of a contaminant following its release in one part of the system, or the decay of a tracer gas in a tracer decay experiment. The introduction of time lags into the model renders the mathematics cumbersome. This note is therefore confined to an analysis of a two-zone model.

2 Theory and analysis

Consider a two-zone model as shown in Figure 1. The zones have volumes V_1 and V_2, and the contaminant concentrations at time t are $x_1(t)$ and $x_2(t)$. The flow, F, from 1 to 2 must, in this simple model, be the same as the flow from 2 to 1, but the time delay on the flow paths may be different. Let the time delay be g_1 for the flow from 1 to 2 and g_2 for the flow from 2 to 1. Assume that at time zero the contaminant concentration in zone 1 is suddenly raised to unity, and everywhere else in the system (including connecting ducts) it is zero. To determine the subsequent pattern of contaminant
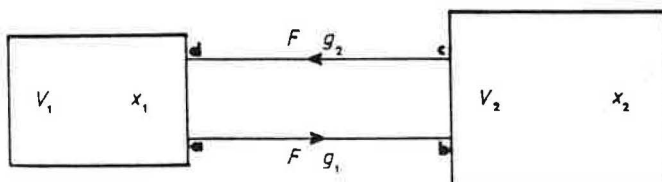

Figure 1   Two-zone air movement model with time lags

concentrations in zones 1 and 2, we may write the following equations:

For $t < g_2$

$$V_1\dot{x}_1(t) = -Fx_1(t) + Fx_2(0)$$

For $t \geq g_2$

$$V_1\dot{x}_1(t) = -Fx_1(t) + Fx_2(t - g_2)$$

For $t < g_1$

$$V_2\dot{x}_2(t) = Fx_1(-0) - Fx_2(t)$$

For $t \geq g_1$

$$V_2\dot{x}_2(t) = Fx_1(t - g_1) - Fx_2(t)$$

The notation $x_1(-0)$ indicates the contaminant concentration in zone 1 before time zero, and is therefore the concentration in the connecting duct ab. Introducing, H, the Heaviside unit step function, and writing $v_1 = V_1/F$, $v_2 = V_2/F$ allows the equations to be written more concisely in the form

$$v_1\dot{x}_1(t) = -x_1(t) + H(t - g_2)x_2(t - g_2)$$
$$- H(t - g_2)x_2(0) + x_2(0)$$
$$v_2\dot{x}_2(t) = H(t - g_1)x_1(t - g_1) - H(t - g_1)x_1(-0)$$
$$+ x_1(-0) - x_2(t)$$

These equations may be solved by routine application of Laplace transform. The transformed equations are

$$v_1(s\bar{x}_1 - x_1(0)) = -\bar{x}_1 + e^{-g_2 s}\bar{x}_2$$
$$- \frac{1}{s}e^{-g_2 s}x_2(0) + \frac{1}{s}x_2(0)$$
$$v_2(s\bar{x}_2 - x_2(0)) = e^{-g_1 s}\bar{x}_1 - \frac{1}{s}e^{-g_1 s}x_1(-0)$$
$$+ \frac{1}{s}x_1(-0) - \bar{x}_2$$

The initial condition is that at $t = 0$, $x_1(0) = 1$, $x_2(0) = x_1(-0) = 0$. The equations therefore simplify to

$$v_1 s\bar{x}_1 - v_1 = -\bar{x}_1 + e^{-g_2 s}\bar{x}_2$$
$$v_2 s\bar{x}_2 = e^{-g_1 s}\bar{x}_1 - \bar{x}_2$$

The solution for $\bar{x}_1$ and $\bar{x}_2$ may be written, with $k = g_1 +$

in the form

$$\bar{x}_1 = \frac{v_1(v_2 s + 1)}{(v_1 s + 1)(v_2 s + 1) - e^{-(g_1 + g_2)s}}$$

$$= \frac{v_1}{(v_1 s + 1)}\left(1 - \frac{e^{-ks}}{(v_1 s + 1)(v_2 s + 1)}\right)^{-1} \quad (1)$$

$$\bar{x}_2 = \frac{v_1 e^{-g_1 s}}{(v_1 s + 1)(v_2 s + 1) - e^{-(g_1 + g_2)s}}$$

$$= \frac{v_1 e^{-g_1 s}}{(v_1 s + 1)(v_2 s + 1)}\left(1 - \frac{e^{-ks}}{(v_1 s + 1)(v_2 s + 1)}\right)^{-1} \quad (2)$$

Using $(1 - a)^{-1} = 1 + a + a^2 + \ldots$, the solutions may be expressed as a series expansion. For example. for $\bar{x}_1$,

$$\bar{x}_1 = \frac{v_1}{(v_1 s + 1)}\left(1 + \frac{e^{-ks}}{(v_1 s + 1)(v_2 s + 1)}\right.$$
$$\left. + \frac{e^{-2ks}}{(v_1 s + 1)^2(v_2 s + 1)^2} + \ldots\right)$$

Similarly for $\bar{x}_2$. To obtain the Laplace inverse for the general case, each term in the series must be split into partial fractions. For instance, provided $v_1 \neq v_2$, the denominator of the second term in $\bar{x}_1$ is

$$\frac{1}{(v_1 s + 1)}\frac{1}{(v_1 s + 1)(v_2 s + 1)} = \frac{v_1/(v_1 - v_2)}{(v_1 s + 1)^2}$$

$$- \frac{v_1 v_2/(v_1 - v_2)^2}{v_1 s + 1} + \frac{v_2^2/(v_1 - v_2)^2}{v_2 s + 1}$$

Hence, taking the Laplace inverse term by term,

$$x_1(t) = e^{-t/v_1} + H(t - k)\left(\frac{t - k}{v_1 - v_2} e^{-(t-k)/v_1}\right.$$
$$\left. - \frac{v_1 v_2}{(v_1 - v_2)^2} e^{-(t-k)/v_1} + \frac{v_1 v_2}{(v_1 - v_2)^2} e^{-(t-k)/v_2}\right)$$
$$+ \text{ terms in } H(t - 2k), H(t - 3k) \text{ etc.}$$

Similarly for $x_2(t)$.

When $v_1 = v_2 = v$, the series expansions for $\bar{x}_1$ and $\bar{x}_2$ may be written as

$$\bar{x}_1 = \frac{1}{(s + v^{-1})} + \frac{e^{-ks}}{v^2(s + v^{-1})^3} + \frac{e^{-2ks}}{v^4(s + v^{-1})^5} + \ldots$$

$$\bar{x}_2 = \frac{e^{-g_1 s}}{v(s + v^{-1})^2} + \frac{e^{-(k+g_1)s}}{v^3(s + v^{-1})^4} + \frac{e^{-(2k+g_1)s}}{v^5(s + v^{-1})^6} + \ldots$$

The Laplace inverse may be obtained directly to give

$$x_1(t) = e^{-t/v}\left(1 + H(t - k)\frac{(t - k)^2}{2!v^2}e^{k/v}\right.$$
$$\left. + H(t - 2k)\frac{(t - 2k)^4}{4!v^4}e^{2k/v} + \ldots\right) \quad (3)$$

$$x_2(t) = e^{-(t-g_1)/v}\left(H(t - g_1)\frac{(t - g_1)}{v}\right.$$
$$+ H(t - \overline{k + g_1})\frac{(t - \overline{k + g_1})^3}{3!v^3}e^{k/v}$$
$$\left. + H(t - \overline{2k + g_1})\frac{(t - \overline{2k + g_1})^5}{5!v^5}e^{2k/v} + \ldots\right) \quad (4)$$

It is interesting to note that when $g_1 = k = 0$, the solution reduces to

$$x_1(t) = \tfrac{1}{2}(1 + e^{-2t/v})$$
$$x_2(t) = \tfrac{1}{2}(1 - e^{-2t/v})$$

which is obviously the solution for the zero time lag case.

The equilibrium value of x_1 and x_2 may be found by means of the final value theorem,

$$\lim_{t \to \infty}[x(t)] = \lim_{s \to 0}[s\bar{x}(s)]$$

Applying this to equations 1 and 2 (with $v_1 = v_2$) gives

$$x_1 = x_2 = \frac{v}{2v + k} \quad \text{at } t = \infty \quad (5)$$

When $k = 0$, the zero time lag case, the equilibrium value is clearly $\tfrac{1}{2}$.

Since v is the time constant of the process, it is convenient to normalise the time dimension with respect to v, i.e.

$$t' = \frac{t}{v}, g_1' = \frac{g_1}{v}, g_2' = \frac{g_2}{v} \text{ and } k' = \frac{k}{v}$$

Assuming plug flow in the connecting ducts, it can then be seen that the volumes of these ducts are given by

$$V_{ab} = g_1 F = g_1' V \text{ and } V_{cd} = g_2 F = g_2' V$$

3 Examples and discussion

Equations 3 and 4 have been evaluated for five examples, with values of k' ranging from zero to 1.2. The zero time lag case, with $k' = 0$, is plotted in Figure 2, and shows the concentrations $x_1(t)$ and $x_2(t)$ smoothly approaching the expected equilibrium value of 0.5. Figures 3 to 6 show the effect of increasing the time lag. For small time lags ($k' = 0.3$ and $k' = 0.6$) the effect is not only to reduce the equilibrium value given by equation 5, but also to cause the concentrations to approach equilibrium more rapidly. For larger time lags ($k' = 0.9$ and $k' = 1.2$), the concentration curves show instability, with an oscillation of half period
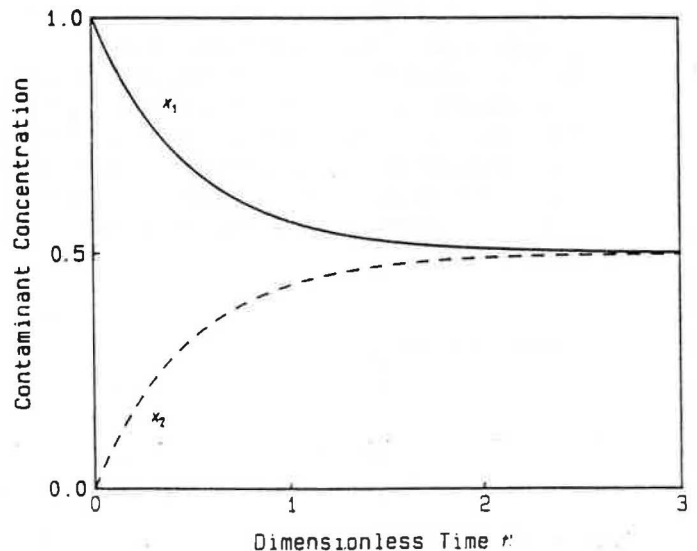

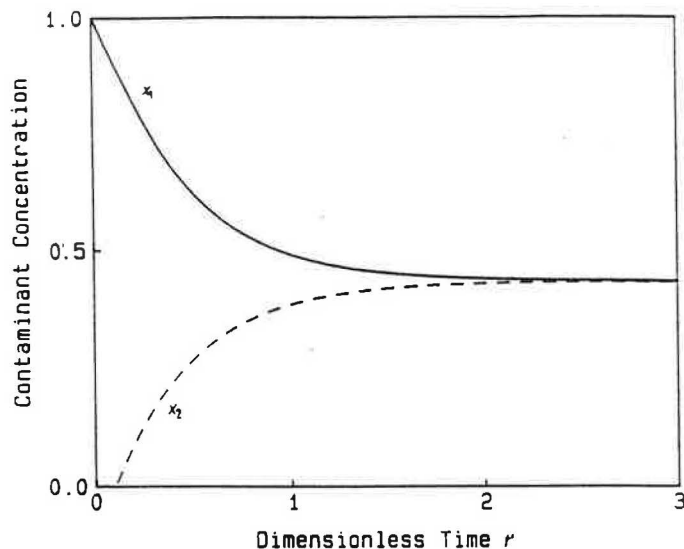**Figure 2** Two-zone decay, zero time lag case. $g_1 = g_2 = 0$
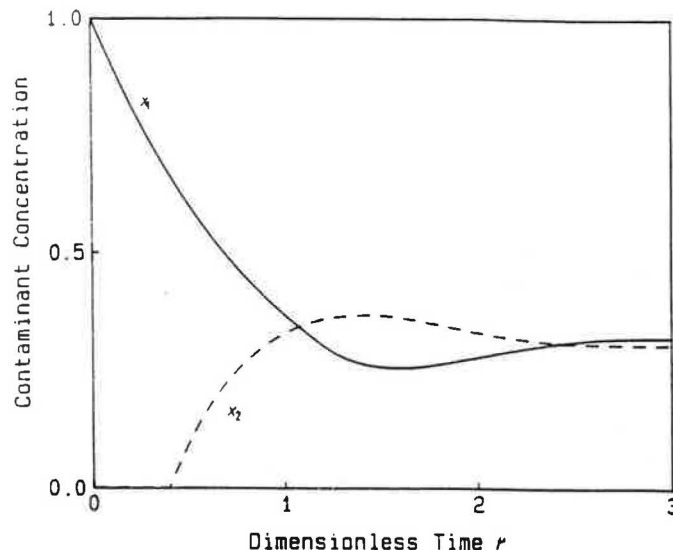
44

Figure 3 Two-zone decay, $k' = 0.3$, $g_1 = 0.1$, $g_2 = 0.2$

Figure 6 Two-zone decay, $k' = 1.2$, $g_1 = 0.4$, $g_2 = 0.8$
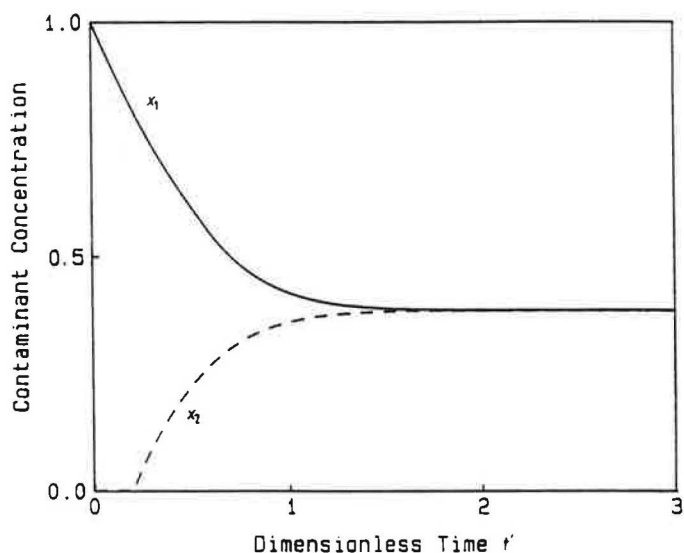

Figure 4 Two-zone decay, $k' = 0.6$, $g_1 = 0.2$, $g_2 = 0.4$
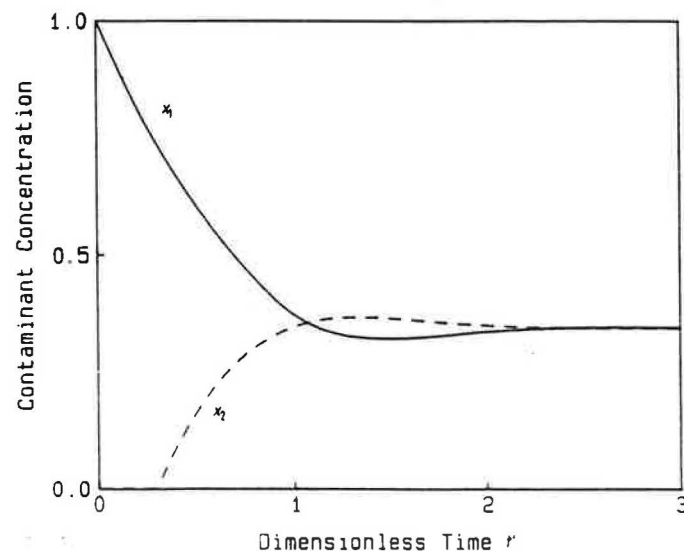

Figure 5 Two-zone decay, $k' = 0.9$, $g_1 = 0.3$, $g_2 = 0.6$

roughly equal to k' and amplitude increasing with k'. The value of k' at which instability begins may be found by looking for a minimum in the $x_1(t)$ curve between $t = k'$ and $t = 2k'$. Taking the first two terms only in equation 3, differentiating, and solving for $(t' - k')$ when $\dot{x}_1(t) = 0$, shows that the required minimum exists when $k' \geqslant \ln 2$, i.e. $k' \geqslant 0.69$. Thus, instability exists when the total volume of the connecting ducts is given by

$$V_{ab} + V_{cd} = (g_1' + g_2')V \geqslant 0.69\,V$$

Inspection of equations 3 and 4 shows that the pattern of the $x_1(t)$ and $x_2(t)$ curves depends only on the total time lag, k', and that the effect of g_1' is to offset the curve for $x_2(t)$ on the time axis. Thus the pattern of the curves, the equilibrium values, and the condition for instability all depend only on k'.

The effect of allowing additional flows between each zone and the outside has not been included in this analysis. Nevertheless it is obvious that if there is no contaminant in the outside air, the effect will be to make $x_1(t)$ and $x_2(t)$ asymptote to zero instead of $1/(2 + k')$, without affecting the periodicity and offset of the decay curves. However, the value of k' at which instability commences may not be the same.

4 Applications

An important application is in the measurement of infiltration rates and air movement by the tracer decay technique. The presence in the measured decay curves of oscillations of the type shown in Figures 5 and 6 would be an indication of the possible existence of time lags. Such oscillatory behaviour has been observed in measurements in industrial buildings, and Figure 7 due to Simons and Waters[5] is an example of measured decay curves in a six-zone building. Clearly, time lags, if present, will affect both the values of the tracer gas concentration and the gradient of the decay curves throughout the decay process.

Estimates of fresh air infiltration based on decay curve gradients may be in error, because compared with normal single-zone and multi-zone theory, gradients will be steeper
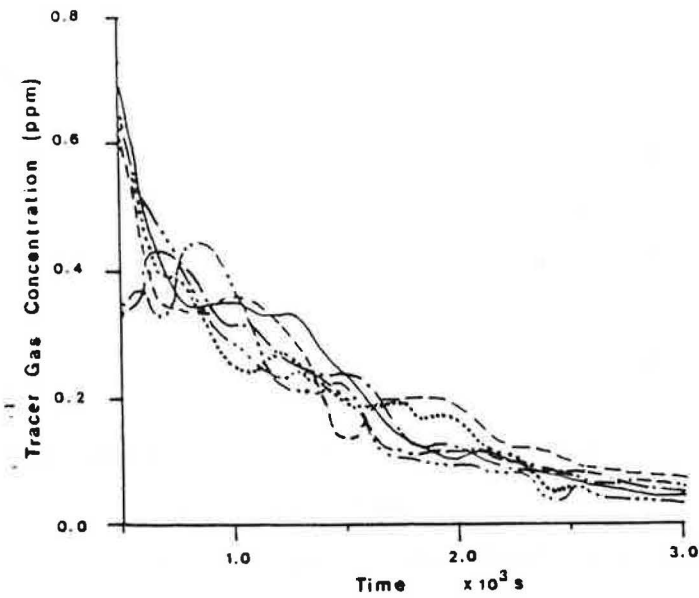
Figure 7 Measured decay curves in a factory building. Zones: 1 ———,
2 —·—·-, 3 ----, 4 —··—··-, 5 ——··—, 6 ········

during the early part of the decay process and shallower
during the later stages.

5 Conclusion

The simple two-zone, model examined in this paper is suf-
ficient to demonstrate that the existence of time delays in
the flow paths between zones has a considerable effect on
the contaminant versus time curves. The method described
here could in principle be extended to predict contaminant
concentrations in models with more than two zones. The
reverse process, of establishing flow rates from measured
tracer decay curves, is clearly more difficult than in the
absence of time lags, as there will be additional unknowns
to evaluate.

References

1 Sinden F W Multi-chamber theory of air infiltration *Building Env.* 13(1)
   21–28 (1978)
2 Sandberg M The multi-chamber theory reconsidered from the viewpoint
   of air quality studies *Building Env.* 19(1) 221–233 (1984)
3 Perera M D A E S Review of techniques for measuring ventilation
   rates in multicelled buildings *Energy Conservation in Buildings—Heating
   Ventilation and Insulation* (ed. Ehringer, Hoyaux and Ziegers) (Dor-
   drecht: Reidl) (1982)
4 Waters J R and Simons M W The evaluation of contaminant con-
   centrations and air flows in a multizone model of a building *Building
   Env.* To be published
5 Simons M W and Waters J R The measurement of ventilation and
   air movement in factory buildings. *Proc. Int. Cong. Building Energy
   Management ICBEM '87, Lausanne* (Sept–Oct 1987)