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Further Consideration of the Height Dependence of Root-Coherence in the Natural Wind

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The growing interest in the response of structures to turbulent wind forces and the realisation of the important role played by root-coherence in the prediction of such response has led to the proposal of several expressions for the power spectrum and the root-coherence function in the natural wind. A more general expression for the power spectral density has evolved and on the basis of it an improved exponential decaying function is put forward for the root-coherence of the longitudinal turbulent component in the natural wind. This takes into account both horizontal and vertical separation between two points.

A modified frequency term is introduced and a power law profile is applied to the decay factor in order to establish the height variation of the root-coherence function.

The consistency of this relationship is investigated by comparison with several sets of empirical data from different sites. The results are encouraging and suggest that this type of approach should be incorporated into dynamic structural response calculations.

INTRODUCTION

RECENTLY there has been a growing interest in the dynamic response of structures due to turbulent wind forces. In order to estimate the total fluctuating load acting on a structure due to the turbulent wind, it is necessary to formulate expressions for the dynamic characteristics of the turbulent wind such as the turbulence intensity, the power spectra, the cross-spectra and so on. In most papers, however, the turbulence in the natural wind is assumed to be homogeneous, in other words, the characteristics of the turbulence are assumed to be independent of position or height[1], although it is widely admitted that the natural wind is almost homogeneous horizontally but not vertically. When the natural mean wind speed profile is established and the standard deviation of the turbulence is considered to be constant with height, the most important problem for wind-loading is how to express the power spectra and the root-coherence (or the cross-spectra) of the longitudinal fluctuating components of the turbulent wind speed.

Some expressions for the power spectra as a function of height have been suggested[2, 3], however, as far as the root-coherence is concerned, the non-homogeneity has hardly been taken into account for reasons of simplicity and convenience in practical applications. One exception to this pattern is the use of a wind speed averaged between two points having some vertical separation[4]. An alternative approach[5] assumes homogeneous isotropic turbulence to exist in the first instance and develops an expression for root-coherence which incorporates some allowance for the variation of the length scale of turbulence with height and yields a complicated function.

It is the aim of this paper to express the root-coherence function more simply but still having dependence on height. This may not be applicable to the general problems of non-homogeneous flow, but simple enough to be manipulated with a combination of a suitable height-dependent power spectral function and a power law for the mean wind speed profile.

The symbols used in the paper are summarised below.

NOMENCLATURE

a, b	= index
B	= width of a structure
$C(z_1, z_2)$	= factor due to z_1 and z_2
C_d	= drag coefficient
C_d^*	= modified drag coefficient
C_M	= mass coefficient
C_M^*	= modified mass coefficient
$\exp(\ )$	= exponential function
$E(\ )$	= average operation
$F(x)$	= function of x
$F_n(t)$	= generalized force of the nth mode
f	= frequency
f^*	= modified frequency
f_n	= natural frequency of the nth mode
H	= height of a structure
i	= $\sqrt{-1}$, indicator of x, y, z
κ, K	= constants
k_H, k_V	= decay factor in the lateral and vertical direction respectively
$\mathbf{K}$	= decay factor matrix

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L_i	= length scale in i-direction where $i = x, y, z$
$\mathscr{L}_1$	= longitudinal length of turbulence (arbitrary value)
m	= mass of a structure per unit area
M_n	= generalized mass of a structure of the nth mode
M'_n	= total generalized mass of the nth mode
n	= modal number
N	= number of degrees of freedom
$P(t)$	= net wind pressure
$p(t)$	= fluctuating component of $P(t)$
r	= a non-dimensional length scale function
r_i	= separation distance, $i = x, y, z$
$\mathbf{r}$	= separation distance vector
$R_u(f)$	= root-coherence function of u
$\mathscr{R}_u(\mathbf{r}, \tau)$	= cross-correlation coefficient of u_1, u_2
$S_u(f)$	= power spectral density function of u
$S_{u_1u_2}(f)$	= cross-spectral density function of u_1, u_2
$S_F(f)$	= power spectral density function of F_n
$S_\delta(f)$	= power spectral density function of δ
t	= time
$U(t)$	= wind speed
$U'(t)$	= relative wind speed
$\bar{U}$	= mean wind speed
$\bar{U}_r$	= reference mean wind speed
$u(t)$	= longitudinal turbulent component of $U(t)$
$\dot{u}(t)$	= wind acceleration
$\sqrt{\overline{u^2}}$	= standard deviation of u
x_1	$= f\mathscr{L}_1/\bar{U}_r$
x	= longitudinal co-ordinate
y, y_1, y_2	= lateral co-ordinates
z, z_1, z_2	= vertical co-ordinates
z_r	= reference height
z_m	= geometric mean of z_1, z_2
α	= power exponent of mean wind speed profile
α'	= power exponent of $C(z_1, z_2)$ profile
α_1	= power exponent of decay factor profile
β	= constant index in power spectral expression
γ	= power exponent of $\mathscr{L}_1$ profile
$\Gamma(\ )$	= gamma function
Δ	= longitudinal displacement
δ	= fluctuating component of Δ
$\dot{\delta}$	= longitudinal velocity
$\ddot{\delta}$	= longitudinal acceleration
$\mu(z)$	= mode shape
ρ	= air mass density
τ	= time lag
ξ	= reduced frequency $fB/\bar{U}$
$\chi_n(f)$	= mechanical admittance of the nth mode
ζ_n	= damping ratio of a structure in the nth mode
ζ'_n	= total damping ratio in the nth mode

REVIEW OF POWER SPECTRAL DENSITY EXPRESSIONS

There have been a great number of measurements of spectra of wind speed in recent years. For higher frequency regions most measurements confirm the Kolmogorov hypothesis[6], however, for lower frequency regions there are still some variations between established formulae for the power spectral density function of the longitudinal turbulent wind speed component u.

Davenport[1, 7] proposed the following height independent expression as the basis of his approach to the prediction of wind induced dynamic structural response.

$$\frac{f \cdot S_u(f)}{\overline{u^2}} = k_1 \frac{x_1^2}{(1+x_1^2)^{4/3}} \tag{1}$$

where $x_1 = f \cdot \mathscr{L}_1/\bar{U}_r$, $f \equiv$ frequency, $\mathscr{L}_1 \equiv$ horizontal length of the wind fluctuation = 1200 m, $k_1 \equiv$ constant $= \frac{2}{3}$ for normalising purposes, $\overline{u^2} \equiv$ variance of u, $\bar{U}_r \equiv$ reference mean wind speed at $z = z_r$ (an arbitrary reference height).

This formula has been used quite often but further investigations suggest that it is slightly conservative for engineering purposes. The first point is that equation (1) has a zero value for $f = 0$Hz, although the one-dimensional spectral density is considered to approach a finite value when the frequency goes to zero[6]. The second point is that equation (1) is independent of height. The horizontal scale $\mathscr{L}_1$ appears to vary from site to site and to increase with height, however, its variation is not clear, in other words, not established at this moment.

In consequence some improved expressions of the power spectral density have been proposed.

According to Hino[2],

$$\frac{f \cdot S_u(f)}{\overline{u^2}} = k_1 \frac{x_1}{(1+x_1^2)^{5/6}} \tag{2}$$

where $x_1 = f \cdot \mathscr{L}_1/\bar{U}_r$, $\mathscr{L}_1 = k_2(z/z_r)^{1-4\alpha}$. k_1 and k_2 are constants with $k_1 = 0.475$ for normalising purposes (see Appendix B). α is a power exponent of the mean wind speed profile. Also a related expression due to Simiu[3] can be rewritten in a similar way to equation (2), as.

$$\frac{f \cdot S_u(f)}{\overline{u^2}} = k_1 \frac{x_1}{(1+x_1)^{5/3}} \tag{3}$$

where $x_1 = f \cdot \mathscr{L}_1/\bar{U}_r$, $\mathscr{L}_1 = k_2(z/z_r)^{1-\alpha}$, k_1 and k_2 are constants with $k_1 = \frac{2}{3}$ for normalising purposes (see Appendix B).

In equation (3), the power exponent α is an equivalent value for the logarithmic mean wind profile used by Simiu. Both equations (2) and (3) satisfy the Kolmogorov hypothesis in the high frequency range just as well as equation (1). Equation (2) has a form suggested by Harris[8], which is known as a Von Karman spectrum, and uses a constant horizontal length of

$$\mathscr{L}_1 = \frac{1}{\sqrt{2}} 1800\,\text{m}.$$

There is not much difference between equation (2) and (3) at lower heights but the variation of $\mathscr{L}_1$ is rather different in each case. Consequently there is a significant difference at greater heights, i.e. at greater values of z.

Equation (2) is derived from the balance of the energy dissipation, assuming a power law profile for the mean wind speed, and equation (3) is based on a logarithmic profile.

Another more general expression can be written as follows[9],

$$\frac{f \cdot S_u(f)}{\overline{u^2}} = k_1 \frac{x_1}{(1+x_1^\beta)^{5/3\beta}} \tag{4}$$

where $x_1 = f \cdot \mathscr{L}_1/\bar{U}_r$, $\mathscr{L}_1 = k_2(z/z_r)^\gamma$, k_1 and k_2 are constants and

$$k_1 = \frac{\beta\Gamma\left(\frac{5}{3\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)\Gamma\left(\frac{2}{3\beta}\right)}$$

(see Appendix B).

According to the U.S. National Bureau of Standards[9], $\beta = 0.845$ in equation (4) is suggested from several sets of empirical data by means of the least squares method. $\beta = 2$ corresponds to equation (2), $\beta = 1$ corresponds to equation (3), and $\beta = \frac{5}{3}$ corresponds to Panofsky–Lumley's expression, which is another well-known formula[10].

More measurements at different sites may indicate other values for β or expressions for $\mathscr{L}_1$. Clearly, there could exist many variations in the form of the power spectral expression depending on site conditions and consequently a general expression of the type shown in equation (4) would be better for engineering purposes. The following discussion is developed on the basis of the power spectral expression given by equation (4).

ROOT-COHERENCE FUNCTION WITH VERTICAL AND HORIZONTAL SEPARATION

One of the well-known expressions for the root-coherence function $R_u(x,f)$, i.e. the cross-correlation coefficient in the frequency domain, has the form of a simple decaying exponential function as suggested by Davenport[1], namely,

$$R_u(x,f) = \exp(-k \cdot fx/\bar{U}_r) \tag{5}$$

where k is a decay constant, x is a separation distance.

This form has been used many times but as in the case of the power spectral expression equation (1), it has been pointed out[4] that equation (5) is slightly conservative.

For the convenience of integration with respect to the surface of a structure the root-coherence function for two points which are both horizontally and vertically separated is expressed as a simple product of the root-coherence functions for the horizontal separation and for the vertical separation in some gust response approaches[7, 11]. However, this simplification causes a significant underestimate in certain circumstances, e.g. up to 20% according to Vickery[12].

A further point is that concerning the height dependency of the root-coherence which is not taken into account in equation (5). An improved expression was suggested by Vickery[4] in the form,

$$R_u(y_2-y_1, z_2-z_1, f) = \exp\left\{-\frac{\sqrt{k_H^2(y_2-y_1)^2 + k_V^2(z_2-z_1)^2}}{\frac{1}{2}[\bar{U}(z_1)+\bar{U}(z_2)]}\cdot f\right\} \tag{6}$$

This assumes that the variation of the decay factors are linear to a wind speed averaged between two points with vertical separation.

Meanwhile, for homogeneous isotropic flow, Harris suggested a theoretical expression for the root-coherence[8] using modified Bessel functions. This has been improved by developing expressions for the root-coherence which incorporate some allowance for the variation of the length scales of turbulence with height[5]. However, this form seems to be rather complicated for practical use but it does indicate that the root-coherence function approaches a value significantly less than unity when the frequency tends to zero, and its value at $f = 0$ varies due to the mean wind speed $\bar{U}_r$ and the horizontal length $\mathscr{L}_1$. These factors could influence computations of dynamic wind force and it is interesting to note that by comparison equations (5) and (6) give root-coherence $= 1$ when $f = 0$.

Considering these points, equation (6) can be improved, as follows,

$$R_u(\mathbf{r}, f) = \exp\left(-\frac{|\mathbf{K}(z_m)\cdot\mathbf{r}|}{\bar{U}_r}\cdot f^*\right) \tag{7}$$

where

$$\mathbf{r} = \begin{Bmatrix} y_2 - y_1 \\ z_2 - z_1 \end{Bmatrix}, \quad \mathbf{K} = \begin{pmatrix} k_H(z_m) & 0 \\ 0 & k_V(z_m) \end{pmatrix}.$$

The decay factors k_H and k_V are assumed to be a function of z_m which is a representative height for the vertical positions z_1 and z_2. Here z_m is introduced as a geometric mean of z_1 and z_2. The reason for using a geometric mean instead of a simple mean is for mathematical convenience since the root of the product of the power spectra is used in the definition of the root-coherence function shown later in equation (9). However, there may not be a significant difference between the geometric mean $\sqrt{z_1 z_2}$ and the simple mean $(z_1+z_2)/2$ in the practical use of the coherence function.

The modified frequency f^* is introduced instead of the frequency f in the root-coherence function for the purpose of consistency and better fit to the empirical data. Referring to Harris's theoretical approach[8],

$$f^* = \sqrt{\frac{\bar{U}_r^2}{\mathscr{L}_1^2(z_m)} + f^2} \tag{8}$$

For the greater values of frequency, i.e.

$$x_1 = \frac{f \cdot \mathscr{L}_1(z_m)}{\bar{U}_r} \gg 1,$$

f^* may be replaced by f without losing accuracy.

In order to formulate the decay factors k_H and k_V as a function of z_m, the cross-correlation coefficient $\mathscr{R}_u(\mathbf{r},\tau)$ can be considered to be related to the root-coherence function. The cross-correlation coefficient is the inverse Fourier transform of the cross-spectral density function from which the root-coherence $R_u(\mathbf{r}, f)$ is defined as,

$$\frac{S_{u_1u_2}(f)}{\overline{u^2}} = R_u(\mathbf{r}, f)\frac{\sqrt{S_{u_1}(f)\cdot S_{u_2}(f)}}{\overline{u^2}} \tag{9}$$

where $S_{u_1u_2}(f)$ is the cross-spectral density function between $u_1 = u(y_1, z_1)$ and $u_2 = u(y_2, z_2)$.

$S_u(f)$ is the power spectral density function of $u(z)$ and the variance $\overline{u^2}$, of the longitudinal gust component $u(z)$ is assumed to be constant with height.

From the Fourier transformation,

$$\mathscr{R}_u(\mathbf{r},\tau) = \int_0^\infty \frac{S_{u_1u_2}(f)}{\overline{u^2}}\cdot e^{i2\pi f\tau}\,df \qquad (10)$$

where $\mathscr{R}_u(\mathbf{r},\tau)$ is the cross-correlation coefficient between u_1 and u_2.

Substituting equation (9) into (10) with $\tau = 0$ for the cross-correlation coefficient $\mathscr{R}_u(\mathbf{r},\tau)$, and assuming that the phase angle or quadrature component of the cross-spectral density can be neglected, equation (11) is obtained.

$$\mathscr{R}_u(\mathbf{r}) = \int_0^\infty R_u(\mathbf{r},f)\cdot\frac{\sqrt{S_{u_1}(f)\cdot S_{u_2}(f)}}{\overline{u^2}}\,df. \qquad (11)$$

Substituting equation (4) for the power spectral density function and equation (7) for the root-coherence function in equation (11) it becomes,

$$\mathscr{R}_u(\mathbf{r}) = \int_0^\infty \exp\left(-\frac{|\mathbf{K}(z_m)\cdot\mathbf{r}|}{\bar{U}_r}\cdot f^*\right) \times \frac{k_1\sqrt{x_1(z_1)\cdot x_1(z_2)}}{f\sqrt{[1+x_1^\beta(z_1)]^{5/3\beta}[1+x_1^\beta(z_2)]^{5/3\beta}}}\,df. \qquad (12)$$

The root product of power spectra in equation (12) is rearranged noting that $x_1 = f\cdot\mathscr{L}_1(z)/\bar{U}_r$ and $\mathscr{L}_1(z) = k_2(z/z_r)^\gamma$,

$$\frac{k_1\sqrt{x_1(z_1)\cdot x_1(z_2)}}{f\sqrt{[1+x_1^\beta(z_1)]^{5/3\beta}[1+x_1^\beta(z_2)]^{5/3\beta}}} = \frac{k_1\sqrt{\mathscr{L}_1(z_1)\cdot\mathscr{L}_1(z_2)}}{\bar{U}_r\sqrt{\left[1+\frac{\mathscr{L}_1^\beta(z_1)}{\mathscr{L}_1^\beta(z_m)}x_1^\beta(z_m)\right]^{5/3\beta}\left[1+\frac{\mathscr{L}_1^\beta(z_2)}{\mathscr{L}_1^\beta(z_m)}x_1^\beta(z_m)\right]^{5/3\beta}}} \qquad (13)$$

Since $\mathscr{L}_1(z_m) = \sqrt{\mathscr{L}_1(z_1)\mathscr{L}_1(z_2)}$ then by letting

$$r = \left[\frac{\mathscr{L}_1(z_2)}{\mathscr{L}_1(z_1)}\right]^{\beta/2} = \left[\frac{\mathscr{L}_1(z_m)}{\mathscr{L}_1(z_1)}\right]^\beta = \left[\frac{\mathscr{L}_1(z_2)}{\mathscr{L}_1(z_m)}\right]^\beta$$

where for $r > 0$ in general $r + 1/r \geqq 2$, the following equations can be obtained.

$$\sqrt{\left[1+\frac{\mathscr{L}_1^\beta(z_1)}{\mathscr{L}_1^\beta(z_m)}x_1^\beta(z_m)\right]^{5/3\beta}\left[1+\frac{\mathscr{L}_1^\beta(z_2)}{\mathscr{L}_1^\beta(z_m)}x_1^\beta(z_m)\right]^{5/3\beta}}$$
$$= \left[1+\left(r+\frac{1}{r}\right)\cdot x_1^\beta(z_m)+x_1^{2\beta}(z_m)\right]^{5/6\beta}$$
$$\geqq [1+2x_1^\beta(z_m)+x_1^{2\beta}(z_m)]^{5/6\beta}$$
$$= [1+x_1^\beta(z_m)]^{5/3\beta} \quad \text{(a lower limit)}$$

and

$$\left[1+\left(r+\frac{1}{r}\right)\cdot x_1^\beta(z_m)+x_1^{2\beta}(z_m)\right]^{5/6\beta}$$
$$\leqq \left[\frac{r+\frac{1}{r}}{2}+\left(r+\frac{1}{r}\right)\cdot x_1^\beta(z_m)+\frac{r+\frac{1}{r}}{2}x_1^{2\beta}(z_m)\right]^{5/6\beta}$$
$$= \left(\frac{r+\frac{1}{r}}{2}\right)^{5/6\beta}[1+x_1^\beta(z_m)]^{5/3\beta} \quad \text{(an upper limit).}$$

Equation (13) can be re-written using a factor $C(z_1,z_2)$ which is defined as follows,

$$\frac{k_1\sqrt{x_1(z_1)\cdot x_1(z_2)}}{f\sqrt{[1+x_1^\beta(z_1)]^{5/3\beta}[1+x_1^\beta(z_2)]^{5/3\beta}}} = \frac{1}{C(z_1,z_2)}\frac{k_1}{\bar{U}_r}\frac{\mathscr{L}_1(z_m)}{[1+x_1^\beta(z_m)]^{5/3\beta}} \qquad (14)$$

where

$$1 \leqq C(z_1,z_2) \leqq \left(\frac{r+\frac{1}{r}}{2}\right)^{5/6\beta}$$
$$= \left\{\frac{\left(\frac{z_m}{z_r}\right)^{\gamma\beta}\left[\left(\frac{z_r}{z_1}\right)^{\gamma\beta}+\left(\frac{z_r}{z_2}\right)^{\gamma\beta}\right]}{2}\right\}^{5/6\beta} \leqq \left(\frac{z_m}{z_r}\right)^{5/6\gamma}$$

and

$$z_2 \geqq \sqrt{z_1 z_2} = z_m \geqq z_1 \geqq z_r > 0.$$

This inequality suggests that $C(z_1,z_2)$ can be expressed simply as a function of the geometric mean height z_m, namely,

$$C(z_1,z_2) = \left(\frac{z_m}{z_r}\right)^{\alpha'} \qquad (15)$$

where $0 \leqq \alpha' \leqq \frac{5}{6}\gamma$. When $z_1 = z_2$, $r = 1$, and so $\alpha' = 0$. It is interesting to note that α' depends on the exponent γ but is independent of the exponent β in the power spectral expression equation (4).

Now, consider the length scales which are defined from the cross-correlation coefficient with no lag time, as follows,

$$L_i = \int_0^\infty \mathscr{R}_u(r_i)\,dr_i \qquad (16)$$

where $i = x, y, z$ and $r_i = i_2 - i_1$.

Generally in three dimensional turbulent flow nine length scales can be defined as combinations of three velocity components and three directions of the separation. However, since the longitudinal velocity component is the major contributor to the fluctuating wind force which causes the along-wind dynamic response of a structure only three length scales out of nine need be taken into account for the longitudinal velocity component.

Clearly $\mathscr{R}_u(r_i)$ is a function of i_1 and i_2 and can be rewritten as a function of position, i.e. the mean of i_1

and i_2 and the difference $r_i = i_2 - i_1$. Since equation (16) has the form of a definite integral with respect to the difference r_i, L_i can be considered as a function of position. If the turbulent flow is assumed to be horizontally homogeneous all correlation coefficients are independent of horizontal position. Then L_x and L_y can be considered constant with horizontal position but L_z may be expressed as a function of vertical position.

In order to obtain the relationship between the decay factors, k_H and k_V, and height the two cases $r_y = 0$ and $r_z = 0$ in equation (12) are discussed.

The integral in equation (16) can be developed from (12), (14) and (15) giving,

$$L_i = \int_0^\infty \int_0^\infty \exp\left(\frac{-k_i(z_m) r_i f^*}{\bar{U}_r}\right) \times \frac{1}{C(z_1, z_2)} \cdot \frac{k_1}{\bar{U}_r} \cdot \frac{\mathcal{L}_1(z_m)}{[1 + x_1^\beta(z_m)]^{5/3\beta}} \mathrm{d}f \mathrm{d}r_i$$

$$= \int_0^\infty \frac{\bar{U}_r}{k_i(z_m) f^*} \left(\frac{z_m}{z_r}\right)^{-\alpha'} \frac{k_1}{\bar{U}_r} \cdot \frac{\mathcal{L}_1(z_m)}{[1 + x_1^\beta(z_m)]^{5/3\beta}} \mathrm{d}f.$$

Since

$$f^* = \sqrt{\frac{\bar{U}_r^2}{\mathcal{L}_1^2(z_m)} + f^2} = \frac{\bar{U}_r}{\mathcal{L}_1(z_m)} \sqrt{1 + x_1^2(z_m)}$$

and

$$\mathrm{d}f = \frac{\bar{U}_r}{\mathcal{L}_1(z_m)} \mathrm{d}x_1.$$

$$L_i = \frac{k_1 \cdot \mathcal{L}_1(z_m)}{k_i(z_m)} \left(\frac{z_m}{z_r}\right)^{-\alpha'} \times \int_0^\infty \frac{\mathrm{d}x_1(z_m)}{[1 + x_1^\beta(z_m)]^{5/3\beta}[1 + x_1^2(z_m)]^{1/2}} \quad (17)$$

where $L_i = L_y, L_z$ and $k_i = k_H, k_V$ and $r_i = r_y, r_z$.

The integral term in equation (17) has a finite constant value. Note that if the frequency f is used instead of the modified frequency f^* for the root-coherence function, the integral does not converge because of the infinite value at $f = 0$.

When the length scale L_i is expressed as a function of height the decay factor can be formulated as a function of height. In homogeneous isotropic flow a relationship between the length scales L_i and the horizontal length $\mathcal{L}_1$ is deduced from Taylor's hypothesis[8], as follows.

$$L_i = K_i \cdot \mathcal{L}_1 \cdot \bar{U}(z)/\bar{U}_r \quad (18)$$

where $K_i = K_x$, K_y and K_z are constants and $K_y = K_z = \frac{1}{2}K_x$.

In the natural wind K_i may not be constant but possibly vary with height. Further measurements have shown that K_i can be expressed as a function of height[5, 15]. However, since the function $K_i(z)$ is not established for different roughness conditions at the present time, K_i is assumed here to be constant, although this may be conservative. If $K_i(z)$ were expressed as an empirical power law function of height, this discussion could be altered easily.

Equating equation (17) to (18),

$$\frac{k_1 \mathcal{L}_1(z_m)}{k_i(z_m)} \left(\frac{z_m}{z_r}\right)^{-\alpha'} \int_0^\infty \frac{\mathrm{d}x_1(z_m)}{[1 + x_1^\beta(z_m)]^{5/3\beta}[1 + x_1^2(z_m)]^{1/2}} = K_i \cdot \mathcal{L}_1(z_m) \cdot \bar{U}(z_m)/\bar{U}_r$$

which yields

$$k_i(z_m) = k_3 \left(\frac{z_m}{z_r}\right)^{-\alpha'} \frac{\bar{U}_r}{\bar{U}(z_m)} = k_3 \left(\frac{z_m}{z_r}\right)^{-\alpha_1} \quad (19)$$

where

$$k_3 = \frac{k_1}{K_i} \int_0^\infty \frac{\mathrm{d}x_1}{(1 + x_1^\beta)^{5/3\beta}(1 + x_1^2)^{1/2}} \quad (19\mathrm{a})$$

and $\alpha_1 = \alpha' + \alpha$. Consequently from equation (15)

$$\alpha \leqq \alpha_1 \leqq \tfrac{5}{6}\gamma + \alpha \quad (19\mathrm{b})$$

and when $z_1 = z_2$, $\alpha_1 = \alpha$. When equation (2) is used as a special case of equation (4) $\gamma = (1 - 4\alpha)$ and so equation (19b) becomes

$$\alpha \leqq \alpha_1 \leqq \tfrac{5}{6}(1 - 4\alpha) + \alpha = \tfrac{5}{6} - \tfrac{7}{3}\alpha. \quad (19\mathrm{c})$$

Alternatively if equation (3) is used as a special case of equation (4), $\gamma = (1 - \alpha)$ and so (19b) becomes

$$\alpha \leqq \alpha_1 \leqq \tfrac{5}{6} + \tfrac{1}{6}\alpha. \quad (19\mathrm{d})$$

The decay factors in the root-coherence function may not have the exact form of a power law expression but the possibility of the existence of such a relation is shown in the above discussion.

By comparing equation (19) with some empirical data measured at different sites the power exponent is estimated in the following section. The constant k_3 in equation (19) may be computed from (19a). However, since parameters k_1 and K_i and β are mostly based on empirical data and have not been established yet, it would appear better to estimate k_3 directly from root-coherence functions obtained from natural wind data.

It can be shown from Harris' work[8] that in homogeneous isotropic flow at a standard reference height $z_r = 10$ m,

$$K_x = 2K_y = 2K_z = 0.118$$

and with $\beta = 2$, $k_1 = 0.475$ [see equation (2)] then $k_3 = 9.01$ (see Appendix B).

COMPARISON WITH EMPIRICAL DATA

Using actual wind speed measurements Shiotani[13], Chuen[14] and Duchène Marullaz[20] computed decay factors for the root-coherence function of the longitudinal components and these are plotted against mean height z_m in Fig. 1: Chuen's results requiring some conversion to make them relative to a standard reference height. Similar decay factors have been deduced here from Harris' natural wind data[8] and are presented in the same figure. Suitable power law exponents for these plots were estimated by means of the least squares method and are listed in Fig. 1, although one of Chuen's results appears spurious and was ignored in calculating the exponent from that set of data.

The range of magnitude of the decay factor varies from one plot to another even though three of the measurement sets were obtained over similar smooth terrain. However, there is close agreement between the values of the exponent, α_1, found from the smooth terrain results, suggesting that the power law expression, equation (19), has some relevance.

It can be seen from Fig. 1 that the decay factor k_3

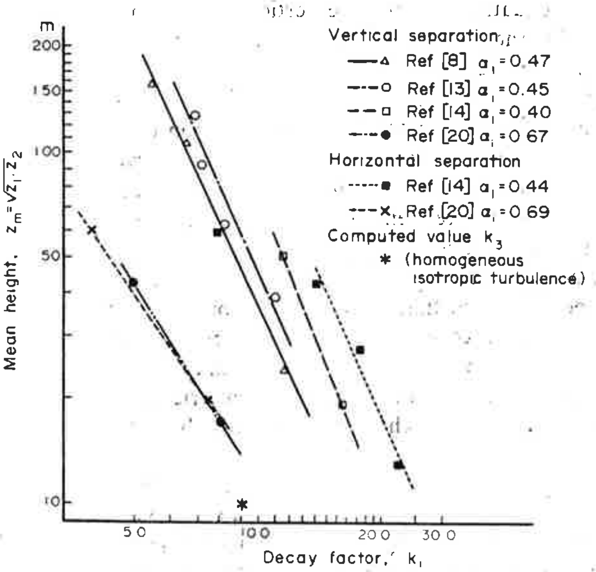

Fig. 1. Variations of decay factor.

computed for the standard reference height $z_r = 10$ m in homogeneous isotropic flow has a lower value than those indicated by extrapolation of the empirical power law curves to the same height.

Such differences could be expected since near to the ground turbulence is not homogeneous or isotropic (see Harris[8]) and under those circumstances the decay factor would be higher.

An interesting feature of Fig. 1 is that the computed value of k_3 for the reference height fits fairly well with the data[20] from the urban location.

It could be anticipated that the lines shown in Fig. 1 would converge on a point at the gradient height where homogeneous isotropic conditions should exist.

The data are not sufficient to form any very definite opinion about the value of α_1 and how it is influenced by the terrain roughness. The smoother surface data in Fig. 1 could be interpreted as converging on a common point at the gradient height. The urban terrain data are much more scant but might suggest that α_1 is greater for increased surface roughness. Since the gradient height increases with ground roughness any points of convergence for rough and smooth data in a plot such as Fig. 1 could not be expected to coincide.

The six values of the exponent α_1 obtained from Fig. 1 are plotted against the corresponding power law exponent α of the mean wind speed profile in Fig. 2

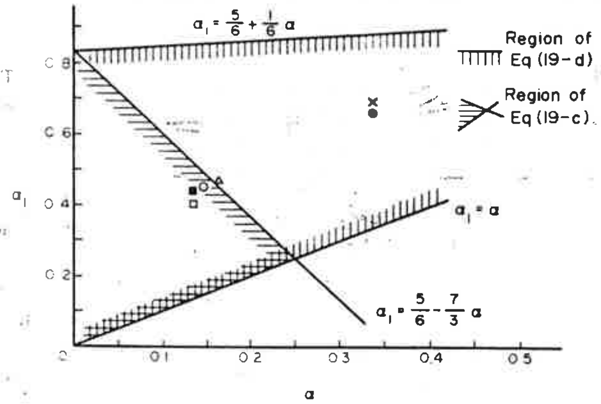

Fig. 2. Power law exponent of decay factor α_1 vs power law exponent of mean windspeed profile α.

and compared with the theoretical region given by equations (19c) and (19d). These equations are based on the power spectral density expressions, equations (2) and (3) respectively, and the upper and lower limits of the α_1-α region could be improved with further experimental information on the relation between L_i and $\mathscr{L}_1$ in equation (18) or by using an alternative form for horizontal length $\mathscr{L}_1$.

The Fig. 2 type of plot should facilitate an understanding of the influence of ground roughness on the decay factor exponent α_1 when more data become available.

A comparison is made in Fig. 3(a), (b) and (c) between the theoretical expression for root-coherence, equations (7) and (19), empirical relations and measured data[8] for three pairs of different heights.

These three plots indicate that the proposed root-coherence expression, equations (7) and (19), is reasonably consistent with measured data and demonstrates its dependence on height.

DISCUSSIONS AND CONCLUSION

There are still not sufficient data available to confirm the consistency of the power spectral expression and the root-coherence expression, especially in a highly built-up area. Both expressions, however, have to be established for the purpose of prediction of the dynamic response of structures. Moreover, it is an important factor for a typical wind-resisting structure like a high rise building in a city centre to represent appropriately the power spectrum and the root-coherence (see Appendix A).

Recent measurements suggest that height dependency is significant for both the power spectrum and root-coherence, which consequently should be taken into account in the representations, with some allowance for the influence of terrain peculiarities. Also the

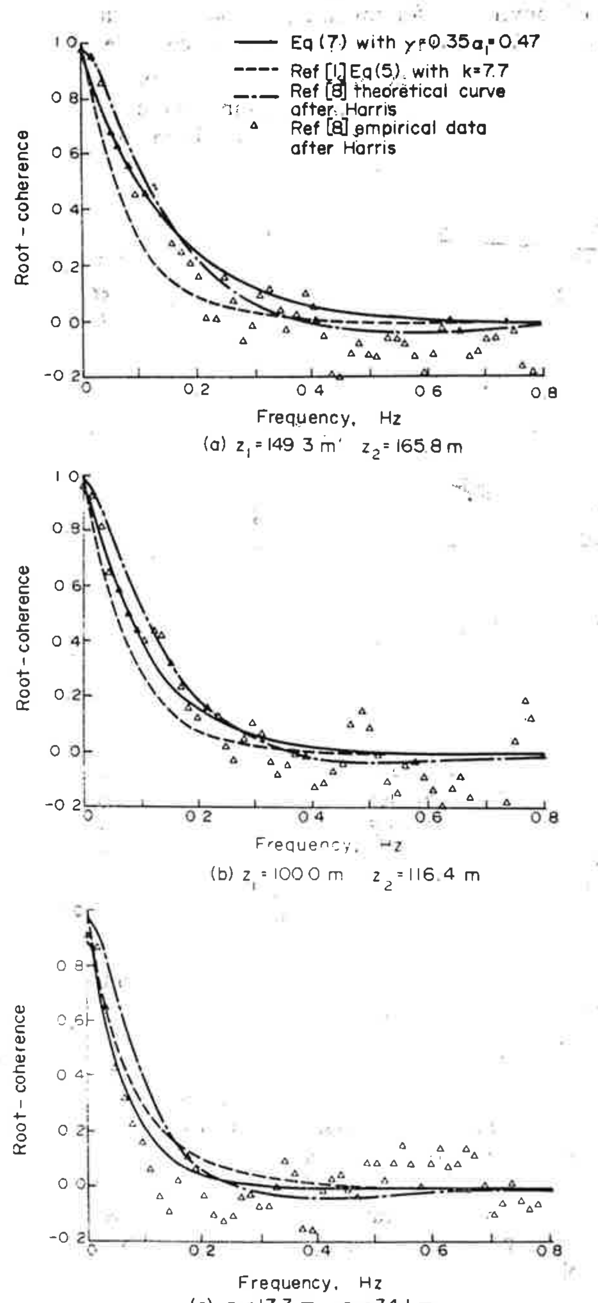

Fig. 3. Comparison between root-coherence functions and empirical data after Harris[8].

variation in value of the root-coherence function at zero frequency can be pointed out from recent measurements—it is less than unity unless the separation distance is zero. Having due regard to the above points, an exponential expression has been developed for the root-coherence function, introducing the modified frequency f^* instead of frequency f and the power law profile with height for the decay factors.

There are two restrictions in the application of the above expressions. Firstly the effect of the quadrature component of the cross-spectra or the phase angle is assumed to be negligible. However, since the natural wind turbulence is not completely homogeneous, the quadrature component exists, even though it may be small. This matter should be investigated in future—considering the effects of the quadrature component on the dynamic wind loading of structures.

Secondly, equation (15) for the vertical separation is only valid for the lower measuring point, given by z_1, above the reference height z_r and z_r should be chosen such that the wind forces below this level do not make a significant contribution to the dynamic response of a structure. This suggests that the power law profile with the exponent α_1, for the vertical decay factor k_V may not hold below the reference height. Further investigation of this point is required using measured data obtained near the ground.

It can be concluded that the expression proposed for the root-coherence of the longitudinal turbulent component between two points with vertical and horizontal separation is consistent with recent empirical data which indicate the height variation of the root-coherence.

The power exponent α_1 for the decay factor in the root-coherence expression is estimated from empirical data and the decay factors, k_H and k_V for horizontal and vertical separation respectively, could be evaluated at the reference height either theoretically or empirically.

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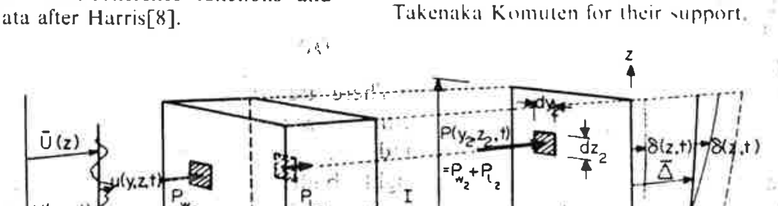

Fig. 4. Wind-structure model.

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APPENDIX A: PREDICTION OF ALONGWIND DYNAMIC RESPONSE OF STRUCTURES IN THE NATURAL WIND

The relationship between the fluctuating drag force and the longitudinal turbulent component of the natural wind is discussed and it is shown how the root-coherence function and the power spectral density of the longitudinal turbulent component contribute to the prediction of the dynamic alongwind response of structures. The theory is a modified form of that developed for a circular cylinder by Cooper and Surry[16].

The assumption of a 'strip theory' relationship between local drag and the local relative velocity for a two-dimensional body[18] can be applied also for a three-dimensional body, by replacing the local drag with the net pressure which is the difference of pressures on the windward and leeward surfaces of a structure. The net pressure $P(y,z,t)$ can be considered to act through the structure on the idealized surface normal to the mean wind direction as shown in Fig. 4.

$$P(y,z,t) = \tfrac{1}{2}\rho C_d(y,z,\xi)U'^2(y,z,t) + \rho B(z)C_M(y,z,\xi)\dot{U}'(y,z,t) \quad \text{(A1)}$$

where C_d is the drag coefficient and C_M is the mass coefficient and both are considered to vary with position and the reduced frequency ξ:

$$\xi = \frac{f\,B(z)}{\bar{U}(z)};$$

$$U'(y,z,t) = U(y,z,t) - \dot{\delta}(y,z,t)$$

is the relative wind speed:

$$U(y,z,t) = \bar{U}(z) + u(y,z,t);$$

$\delta(y,z,t)$ is the longitudinal component of structure dynamic displacement; $\dot{\delta}(y,z,t)$ is the structure velocity; $B(z)$ is the width of a structure; and, ρ is the air mass density.

If the turbulence and the fluctuating motion of the structure are both small, the second order terms in u and $\dot{\delta}$ can be neglected. Then equation (A1) can be written as,

$$\begin{aligned} P(y,z,t) = {} & \tfrac{1}{2}\rho C_d(y,z,0)\bar{U}^2(z) \\ & + \rho C_d(y,z,\xi)\bar{U}(z)u(y,z,t) \\ & + \rho B(z)C_M(y,z,\xi)\dot{u}(y,z,t) \\ & - \rho C_d(y,z,\xi)\bar{U}(z)\dot{\delta}(y,z,t) \\ & - \rho B(z)C_M(y,z,\xi)\ddot{\delta}(y,z,t). \end{aligned} \quad \text{(A2)}$$

The last two terms of equation (A2) are not dependent on the turbulence. Consequently they are considered as the additional damping and mass in the equation of motion of the structure. Then the dynamic part of the drag net pressure $p(y,z,t)$ can be written as,

$$p(y,z,t) = \rho C_d(y,z,\xi)u(y,z,t)\bar{U}(z) + \rho B(z)C_M(y,z,\xi)\dot{u}(y,z,t) \quad \text{(A3)}$$

The dynamic displacement response $\delta(y,z,t)$ is a random function made up of components from the various independent modes of vibration. It is treated by a statistical approach relating the power spectral density of $\delta(y,z,t)$ and the power spectral density of generalised total dynamic force $F(t)$, i.e. the dynamic response of a structure can be determined by solving the normal equations of motion[19], taking the aerodynamic damping and mass terms mentioned above into account.

$$\ddot{q}_n(t)+2\zeta_n'(2\pi f_n)\dot{q}_n(t)+(2\pi f_n)^2q_n(t)=\frac{F_n(t)}{M_n'} \quad \text{(A4)}$$

where q_n is the generalised displacement of the nth mode:

$$\delta(y,z,t)=\delta(z,t)=\sum_{n=1}^{N}\mu_n(z)\cdot q_n(t)$$

assuming the structural displacement to be uniform over its width: $\mu_n(z)$ is the nth mode shape; f_n is the nth natural frequency;

$$M_n'=\int_0^H\int_0^{B(z)}[m(z)+\rho B(z)C_M(y,z,\xi)]\mu_n^2(z)\,\mathrm{d}y\,\mathrm{d}z;$$

$m(z)$ is the mass of structure per unit surface area:

$$\zeta_n'=\zeta_n+\int_0^H\int_0^{B(z)}\rho\frac{C_d(y,z,\xi)\bar{U}(z)\mu_n^2(z)}{2M_n'(2\pi f_n)}\mathrm{d}y\,\mathrm{d}z;$$

ζ_n is the critical damping ratio of the structure of nth mode; and, $F_n(t)$ is the generalised force associated with the turbulence.

$$F_n(t)=\int_0^H\int_0^{B(z)}p(y,z,t)\mu_n(z)\,\mathrm{d}y\,\mathrm{d}z. \quad \text{(A5)}$$

For the most lightly damped structures the cross-coupling between modes is unlikely. Therefore the power spectral density of the response $S_\delta(f)$ can be written as follows as a solution of equation (A4),

$$S_\delta(f)=\sum_{n=1}^{N}\mu_n^2(z)|\chi_n(f)|^2S_{F_n}(f) \quad \text{(A6)}$$

where

$$|\chi_n(f)|^2=\frac{1}{(4\pi^2M_n')^2[f^4+f_n^4+(4\zeta_n'^2-2)f^2f_n^2]}$$

Now the generalised force of the nth mode $F_n(t)$ can be computed by substituting equation (A3) into (A5).

$$F_n(t)=\int_0^H\int_0^{B(z)}\{\rho C_d(y,z,\xi)\bar{U}(z)u(y,z,t)+\rho C_M(y,z,\xi)B(z)\dot{u}(y,z,t)\}\mu_n(z)\,\mathrm{d}y\,\mathrm{d}z. \quad \text{(A7)}$$

In equation (A7), $F_n(t)$ is expressed as a function of t and ξ, but since for lightly damped structures only the components of response in the narrow band of frequency around resonance in a particular mode will be of significance, therefore only the corresponding components of C_d and C_M need be taken into account. Then equation (A7) can be modified accordingly.

$$F_n(t)=\int_0^H\int_0^{B(z)}\{C_{d_n}^*(y,z)u(y,z,t)+C_{M_n}^*(y,z)\dot{u}(y,z,t)\}\,\mathrm{d}y\,\mathrm{d}z \quad \text{(A8)}$$

where

$$C_{d_n}^*(y,z)=\rho C_d(y,z,\xi_n)\bar{U}(z)\mu_n(z),$$

$$C_{M_n}^*(y,z)=\rho C_M(y,z,\xi_n)B(z)\mu_n(z),$$

$$\xi_n=\frac{f_nB(z)}{\bar{U}(z)}$$

Now $S_{F_n}(f)$ can be defined as a Fourier transform of the auto-correlation function of $F_n(t)$ as follows,

$$S_{F_n}(f)=2\int_{-\infty}^{\infty}\mathscr{R}_{F_n}(\tau)\mathrm{e}^{-i2\pi f\tau}\mathrm{d}\tau \quad \text{(A9)}$$

where

$$\mathscr{R}_{F_n}(\tau)=E[F_n(t)\cdot F_n(t+\tau)]. \quad \text{(A10)}$$

Substituting equation (A8) into (A10) and equation (A10) into (A9), the power spectral density function of generalised force of the nth mode S_{F_n} can be computed[17].

$$\begin{aligned}S_{F_n}(f)=2\int_{-\infty}^{\infty}\int_0^H\int_0^H\int_0^{B(z)}\int_0^{B(z)}&\mathscr{R}_{u_1,u_2}(\tau)\\&\times\{C_{d_n}^*(y_1,z_1)C_{d_n}^*(y_2,z_2)+2\pi ifC_{d_n}^*(y_1,z_1)\\&\times C_{M_n}^*(y_2,z_2)-2\pi ifC_{M_n}^*(y_1,z_1)C_{d_n}^*(y_2,z_2)\\&+4\pi^2f^2C_{M_n}^*(y_1,z_1)C_{M_n}^*(y_2,z_2)\}\,\mathrm{d}y_1\,\mathrm{d}y_2\\&\times\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{e}^{-i2\pi f\tau}\mathrm{d}\tau. \quad \text{(A11)}\end{aligned}$$

Since the cross-spectrum of the turbulent component is given as

$$S_{u_1,u_2}(f)=2\int_{-\infty}^{\infty}\mathscr{R}_{u_1,u_2}(\tau)\mathrm{e}^{-i2\pi f\tau}\mathrm{d}\tau \quad \text{(A12)}$$

equation (A11) can be rearranged using equation (A12), namely,

$$\begin{aligned}S_{F_n}(f)=\int_0^H\int_0^H\int_0^{B(z)}\int_0^{B(z)}&S_{u_1,u_2}(f)\\&\times\{C_{d_n}^*(y_1,z_1)C_{d_n}^*(y_2,z_2)+i2\pi f\\&\times[C_{d_n}^*(y_1,z_1)C_{M_n}^*(y_2,z_2)-C_{M_n}^*(y_1,z_1)C_{d_n}^*(y_2,z_2)]\\&+4\pi^2f^2C_{M_n}^*(y_1,z_1)C_{M_n}^*(y_2,z_2)\}\\&\times\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}z_1\,\mathrm{d}z_2. \quad \text{(A13)}\end{aligned}$$

If C_d and C_M are assumed to be constant with y and have the same profile with z,

$$C_{d_n}^*(z_1)C_{M_n}^*(z_2)-C_{M_n}^*(z_1)C_{d_n}^*(z_2)=0.$$

Consequently equation (A13) can be simplified as,

$$\begin{aligned}S_{F_n}(f)=\int_0^H\int_0^H\int_0^{B(z)}\int_0^{B(z)}&S_{u_1,u_2}(f)\\&\times\{C_{d_n}^*(z_1)C_{d_n}^*(z_2)+4\pi^2f^2C_{M_n}^*(z_1)C_{M_n}^*(z_2)\}\\&\cdot\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}z_1\,\mathrm{d}z_2. \quad \text{(A14)}\end{aligned}$$

Generally the natural wind is not a homogeneous turbulent flow and so the cross-spectral density function consists of real and imaginary parts. However, since the power spectrum of the generalised force is a real function the real part of the cross-spectrum (i.e. the co-spectrum) can be taken into account instead of $S_{u_1,u_2}(f)$ in equation (A14).

For further simplification, if the imaginary part or quadrature component of the cross-spectrum of longitudinal turbulence is assumed to be negligible, the root-coherence becomes identical with the normalised co-spectrum, i.e.

$$\begin{aligned}S_{F_n}(f)=\int_0^H\int_0^H\int_0^{B(z)}\int_0^{B(z)}&R_{u_1u_2}(f)\sqrt{S_{u_1}(f)S_{u_2}(f)}\\&\times\{C_{d_n}^*(z_1)C_{d_n}^*(z_2)+4\pi^2f^2C_{M_n}^*(z_1)C_{M_n}^*(z_2)\}\\&\times\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}z_1\,\mathrm{d}z_2. \quad \text{(A15)}\end{aligned}$$

Then the variance of the dynamic response can be obtained as follows,

$$\overline{\delta^2}(z) = \int_0^\infty S_\delta(f)\,df$$

and from equation (A6),

$$\overline{\delta^2}(z) = \int_0^\infty \sum_{n=1}^{\infty} \mu_n^2(z)|\chi_n(f)|^2 S_{F_n}(f)\,df. \quad (A16)$$

For a tall building which has its fundamental natural frequency much greater than the peak frequency of the power spectrum of wind turbulence it should be sufficient to consider only the first two frequency modes of vibration, i.e. $N = 2$.

The instantaneous maximum value of the dynamic response can be obtained from the mean displacement and the r.m.s. value in terms of a peak factor g; which depends on a probability distribution[7], as.

$$\Delta_{max} = \bar{\Delta} + g.\sqrt{\overline{\delta^2}}. \quad (A17)$$

APPENDIX B: INTEGRATION OF A POWER SPECTRAL DENSITY FUNCTION

All power spectral expressions referred to here—equations (1-4)—can be integrated with respect to the frequency from 0 to infinity and the value of the definite integral becomes unity when those power spectral expressions are normalised by the variance of the turbulent component. Then each constant k_1 in those equations can be obtained. For

$$F(x) = \frac{1}{(1+x^a)^b} \quad (B1)$$

the definite integral from $f = 0$ to infinity can be computed as follows. Let $X = x^a$, then

$$dX = ax^{a-1}\,dx = aX^{(a-1)/a}\,dx.$$

Consequently,

$$\int_0^\infty F(x)\,dx = \int_0^\infty \frac{dx}{(1+x^a)^b} = \int_0^\infty \frac{X^{-(a-1)/a}}{a(1+X)^b}\,dX. \quad (B2)$$

Since it is known that

$$\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (B3)$$

where $\Gamma(\ )$ is a gamma function, from equation (B2)

$$n-1 = -\left(\frac{a-1}{a}\right), \quad m+n = b$$

which yields

$$n = \frac{1}{a}, \quad m = b - \frac{1}{a}.$$

Then equation (B2) becomes

$$\int_0^\infty F(x)\,dx = \frac{\Gamma\left(\frac{1}{a}\right)\Gamma\left(b-\frac{1}{a}\right)}{a\Gamma(b)} \quad (B4)$$

In the case of equation (1) a direct integration is possible without resort to gamma functions since it can be rewritten as,

$$\frac{S_u(f)}{\overline{u^2}} = \frac{k_1\mathscr{L}_1}{\bar{U}_r}\cdot\frac{x_1}{(1+x_1^2)^{4/3}}$$

and so,

$$\frac{1}{\overline{u^2}}\int_0^\infty S(f)\,df = 1 = \frac{k_1\mathscr{L}_1}{\bar{U}_2}\int_0^\infty \frac{x_1}{(1+x_1^2)^{4/3}}\,df$$

$$= k_1\int_0^\infty \frac{x_1}{(1+x_1^2)^{4/3}}\,dx_1 = \frac{k_1}{2}\int_0^\infty \frac{dX}{X^{4/3}} = \frac{3}{2}\cdot k_1$$

hence

$$k_1 = \frac{2}{3}. \quad (B5)$$

By contrast equation (4) is in the form,

$$\frac{S_u(f)}{\overline{u^2}} = \frac{k_1\mathscr{L}_1}{\bar{U}_r}\cdot\frac{1}{(1+x_1^\beta)^{5/3\beta}}$$

and

$$\frac{1}{\overline{u^2}}\int_0^\infty S(f)\,df = 1 = \frac{k_1\mathscr{L}_1}{\bar{U}_r}\int_0^\infty \frac{1}{(1+x_1^\beta)^{5/3\beta}}\,df$$

$$= k_1\int_0^\infty \frac{dx_1}{(1+x_1^\beta)^{5/3\beta}}.$$

Consequently, by comparison with equation (B1)

$$k_1\int_0^\infty F(x_1)\,dx_1 = 1$$

where $a = \beta$ and $b = 5/3\beta$. Then from equation (B4)

$$k_1 = \frac{\beta\Gamma\left(\frac{5}{3\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)\Gamma\left(\frac{2}{3\beta}\right)}. \quad (B6)$$

The values k_1 corresponding to the expressions in equations (2–4) are obtained from equation (B6) and are summarised in Table B1 together with the value of k_1 appropriate to equation (1) which is given by equation (B5). Similarly the integral in equation (19a) can be evaluated for $\beta = 2$, i.e.

$$\int_0^\infty \frac{dx_1}{(1+x_1^2)^{5/6}(1+x_1^2)^{1/2}} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{6})}{2\Gamma(\frac{4}{3})} = 1.119 \quad (B7)$$

which is required in order to establish k_3.

Table B1

Equation	Source	Power exponent β	k_1
(1)	Davenport[1]		2/3
(2)	Harris[8] and Hino[2]	2	0·475
(3)	Simiu[3]	1	2/3
(4)	National Bureau of Standards[9]	0·845	0·769
	Panofsky and Lumley[10]	5/3	0·505