Acc 1313

1895

A 008

Summary Large, multicelled and naturally ventilated buildings pose many inherent problems for the measurement of overall infiltration rates using tracer gases. Considering a single tracer gas decay technique, the most obvious problems are: (a) local variations in infiltration, (b) imperfect internal mixing of the air, and (c) practical difficulties in distributing (i.e. seeding) the tracer gas and subsequently obtaining air samples. This paper proposes a relatively simple technique which avoids these problems and which, if successful, makes a breakthrough in the measurement of infiltration rates in large and complex buildings. By considering a multicell model, it is shown that it can be sufficient to seed part of a building with a single tracer gas in order to measure the overall infiltration rate to a good approximation.

Strategy for measuring infiltration rates in large, multicelled and naturally ventilated buildings using a single tracer gas

M. D. A. E. S. PERERA, BSc(Eng), PhD, CEng, MRAeS, FRMetS and R. R. WALKER, BSc, MSc, DIC, FRMetS

List of symbols

a_{ij}	elements of square matrix [A]
[A]	square matrix containing constant coefficients
$C(t)$	tracer concentration (by volume) at time t
$\dot{C}(t)$	time derivative of concentration
$\{C\}$	column vector whose elements are the tracer concentrations in each cell
$\{\dot{C}\}$	column vector whose elements are the time derivatives of the tracer concentrations in each cell
m_1, m_2	intercell airflows normalised by volume of zone of origin
Q_T	'total' infiltration flow into whole building
Q_{ij}	volume flow rate from cell i (= 0, 1, 2, ..) to cell j (= 0, 1, 2, ..) where the zeroth cell refers to the outside
[Q]	square matrix whose elements are the intercell airflows Q_{ij}
R	'whole building' infiltration rate
r_1, r_2	exfiltration rates of the individual cells
V_i	volume of cell i
[V]	square matrix containing volumes of cells
$[V]^{-1}$	inverse of matrix [V]
β_1, β_2, β_3	variable superimposed flows, normalised by the total infiltration Q_T
λ	eigenvalues of the matrix $[V]^{-1}[Q]$

1 Introduction

Although most buildings, whether commercial, public or domestic, rely on natural ventilation, its prediction is one of the most difficult aspects of building design. In recent years, research into air infiltration and ventilation has increased, but mainly with respect to dwellings rather than more complex buildings like offices. Problems of scale and lack of appropriate techniques have deterred investigations of these bigger, multicelled buildings.

A previous report[1] described possible measurement techniques that could be used in multicelled buildings and laid down the theoretical basis underlying the techniques. Subsequently, interzonal airflows in two large buildings were measured[2,3] using multiple tracer gases. However, the cost and complexity of carrying out multiple tracer experiments can be prohibitive and in many instances it would be beneficial to obtain less comprehensive, but nevertheless useful information using only a single tracer gas.

Large, multicelled and naturally ventilated buildings pose many inherent problems for the measurement of overall infiltration rates using a single tracer gas. Considering a single tracer gas decay technique, the most obvious problems are:

(a) local variations in infiltration,
(b) imperfect internal mixing of the air, and
(c) practical difficulties in distributing (i.e. seeding) the tracer gas and subsequently obtaining air samples.

This paper proposes a relatively simple technique which avoids these problems. It is illustrated by analysing a two-cell model and then by carrying out a case study of a five-zone representation of a real building. The results suggest that it is sufficient to seed part of a building with a single tracer gas in order to measure the whole building infiltration rate to a good approximation.

2 Theory

In a well-mixed single cell building, the single eigenvalue solution to the tracer decay equation (Equation 3 given below), represents the air change rate of that building. What is important in this paper is, however, whether the eigenvalue solutions obtained by considering a multicelled building can in any way be used as a measure of the air change rate of the whole building.

If a tracer gas, at an initial concentration level of $\{C(0)\}$, is allowed to decay in a multicelled building, then its subsequent behaviour at any later time, t, is governed[1] by the system of ordinary differential equations

$$\{\dot{C}\} = [V]^{-1}[Q].\{C\} \tag{1}$$

where [V] is the cell-volume matrix and [Q] describes the intercell airflows.

It can be proven[4] that there is a unique solution:

$$\{C(t)\} = \exp([V]^{-1}[Q]t)\{C(0)\} \tag{2}$$

satisfying Equation (1).

The authors are with the Building Research Establishment, Garston, Watford, Herts.

The paper was first received on 7 January 1985, and in revised form on 16 March 1985.

, Vol 6 No 2, 1985

It can be further shown[4] that if the matrix $[V]^{-1}.[Q]$ has distinct eigenvalues λ, then the solution reduces to:

$$\{C(t)\} = [A]\,\{e^{\lambda t}\} \qquad (3)$$

where $[A]$ is dependent on $\{C(O)\}$ but is independent of time t. It can therefore be seen that each element of $\{C(t)\}$ is a linear combination of exponential functions with no polynomial contributions.

In this Section, a two-cell model is considered in order to obtain a feel for the problem. Following this, the study is extended to an N-cell model where the N eigenvalues are numerically computed.

2.1 Two cell model

Let V_1 and V_2 be the volumes of two interconnected cells, 1 and 2 and let Q_{ij} be the airflow from cell i (= 0, 1, 2) to cell j (= 0, 1, 2) where the outside air is considered as cell 0. The two eigenvalues are then given (Appendix) by,

$$-2\lambda = \left(\frac{Q_{01}}{V_1} + \frac{Q_{02}}{V_2}\right) + \left(\frac{m_2V_2}{V_1} + \frac{m_1V_1}{V_2}\right) \pm \sqrt{\left[\left\{\left(\frac{Q_{01}}{V_1} - \frac{Q_{02}}{V_2}\right) + \left(\frac{m_2V_2}{V_1} - \frac{m_1V_1}{V_2}\right)\right\}^2 + 4m_1m_2\right]} \qquad (4)$$

where $m_i = Q_{ij}/V_i$ for j = 1, 2

Taking the negative square root term, the minimum eigenvalue is obtained. The square root term in the resulting equation consists of an infiltration term,

$$X = (Q_{01}/V_1) - (Q_{02}/V_2)$$

and mixing terms. If the infiltration rates in each cell are approximately similar to each other, or if the mixing terms are large in comparison with X, then the square root term can be expanded as a Maclaurin series in X for small X.

Therefore by expressing Equation (4) in terms of a polynomial in X for small X, and then by,

(a) collecting like terms; and
(b) replacing m_2 by considering the airflows into Cell 1,

Equation (4) can be simplified to,

$$-\lambda_{min} = \frac{\dfrac{m_1Q_{01}}{V_2} + \left(\dfrac{Q_{10} - Q_{01} + m_1V_1}{V_2}\right)\dfrac{Q_{02}}{V_1}}{\dfrac{m_1V_1}{V_2} + \dfrac{Q_{10} - Q_{01} + m_1V_1}{V_1}} + O(X^2) + \ldots \qquad (5)$$

If the following hold true,

(a) there is enhanced mixing between cells, i.e. m large, or,
(b) infiltration balances exfiltration in each cell,

the Equation (5) can be further reduced to:

$$-\lambda_{min} = (Q_{01} + Q_{02})/(V_1 + V_2) \qquad (6)$$
$$= R$$

where R is the total infiltration rate.

It might be concluded that the conditions required for the dominant eigenvalue λ_{min} to tend towards the infiltration rate R, are rather restrictive. It is suggested, however, that the two-cell model is an extreme simplification of a multicelled building, and that in a real building two factors help towards correspondence between λ_{min} and R; namely

(a) the influence of any one cell is reduced since there are many cells, and
(b) there may be a combination of internal mixing, equal local infiltration rates and locally balanced infiltration and exfiltration.

3 Case study

The eigenvalues for an N-cell model can, in theory, be obtained by solving an Nth degree polynomial. An analytical solution for a two-cell model is easy to obtain but as the degree of the polynomial increases to three or four, such a solution becomes more difficult. No general solution exists for fifth and higher degree polynomials. Recourse then has to be made to numerical computation.

Equation 3 for an N-cell model can be rewritten as,

$$C_i(t) = \sum_{j=1}^{N} a_{ij}\exp(\lambda_j t) \qquad (9)$$

A computer program was written to evaluate the coefficients a_{ij} and the eigenvalues λ_j. This has been validated by comparison with full scale data[2,3].

3.1 Modelling the building

For the case study, a five zone model representation of a full-scale building is considered. The building is a three-storey 'low energy' office block[5] which is tightly constructed in order to reduce infiltration heat losses. It is rectangular in plan (60 m × 12 m), three storeys high (floor to ceiling height of each floor is 2.6 m) and has its major axis aligned east-west. The usable volume of this building is taken to be 5286 m³.

This building can notionally be divided by its two stairwells into a centre section and an east and west wing. The west staircase provides access from the main entrance in the ground floor level to all other floors. On each floor and in each wing, offices are located along either side (north and south) of a central corridor.

For the purpose of tracer gas seeding, and following earlier (as yet unreported) experimental work, the building was taken to consist of the five zones shown in Table 1.

Table 1. Details of building.

Zone No.	Description	Vol. (m³)	Vol. ratio
1	Rooms in central section of 2nd floor	574	0.109
2	Corridor in central section of 2nd floor	86	0.016
3	Ceiling void above Zone 2	29	0.005
4	West stairwell	170	0.033
5	Remainder of the building	4,427	0.837

The volume ratios were calculated by dividing the zone volumes with the total usable volume of the building. A zone in the present context may be a single cell within which mixing is assumed perfect or a collection of such cells which can be taken to act as one aggregate cell.

3.2 Modelling the airflows

During the heating season, the building is mechanically ventilated. Using multiple tracer gases, interzone airflows have been measured[3] with the ventilation system in operation.

The mechanical ventilation system is switched off during the summer months. In this mode, and with all external

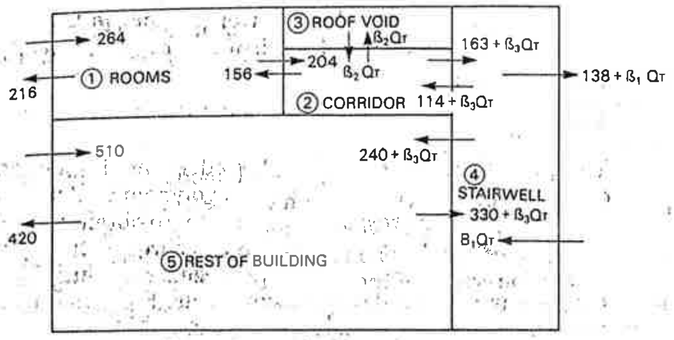

Fig. 1. Airflows (m^3/h) in building—(building drawn not to scale).

doors and windows closed, preliminary measurements were made to determine the interzonal airflows and infiltration rates. Results obtained from these measurements were used (Fig. 1) to provide representative flow rates for use in this computer study.

When measurements to determine infiltration rates are carried out, it is probable that certain zones may be over- or under-ventilated with respect to the rest of the building. To assess these, as well as to determine the effect of using 'mixing' fans to enhance dispersion of tracer gas within the building, various airflows (denoted by β) were superimposed on the airflow pattern in Fig. 1. They were configured so as to maintain the volume-flow balance within the model. The cases shown in Table 2 were considered:

Table 2. β **values.**

BETA	Value	To represent effect of:
β_1	0.0, 0.2, 0.4	over-ventilated zone
β_2	0.0, 0.1	stagnant or 'dead' space
β_3	0.0, 0.2, 0.4	enhanced internal mixing

Table 3. Details of computer 'runs' used in case study.

	Superimposed flows			Eigenvalues					
Run	β_1	β_2	β_3	λ_1	λ_2	λ_3	λ_4	λ_5	Comment
1	0.0	0.0	0.0	31.6	15.2	3.37	0.840	0.0	Standard
2	0.0	0.0	0.2	52.0	19.2	3.80	0.859	0.0	Enhanced mixing
3	0.0	0.0	0.4	72.8	23.0	4.06	0.870	0.0	Enhanced mixing
4	0.0	0.1	0.0	41.3	12.2	3.28	0.839	19.7	Void usage
5	0.2	0.0	0.0	27.5	15.8	2.75	0.756	0.0	Stairwell over-ventilated
6	0.4	0.0	0.0	25.8	14.2	2.09	0.613	0.0	Stairwell over-ventilated

Notes:
(1) β_1, β_2, β_3 are superimposed flows, and represent the following:
β_1 . . . effect of stairwell over-ventilation
β_2 . . . effect of 'void' usage
β_3 . . . effect of 'enhanced' mixing
(2) λ_4 is the dominant eigenvalue.
(3) Whole building infiltration rate, R, is 1.

4 Results and Discussion

Table 3 contains details and results obtained from some of the computer 'runs' which support the discussion that follows. In all of these, only Zone 1 (i.e. rooms) was seeded with a tracer. In all cases, airflows (Fig. 1) were normalised by the total fresh air flow into the building, Q_T, and volume ratios were used.

4.1 Eigenvalues

Hernandez and Ring[6] have pointed out that, if there is no connection between cells, then each of the eigenvalues obtained in a multicell solution could, in theory, be identified with one of the cells. This was seen earlier in the two-cell model and it occurs because, with zero intercell mixing, tracer gases will decay at a distinct rate, independent of the other cells. This 'association' of each eigenvalue with a cell is not strictly possible when the cells are well-connected.

However, using results from Run 4 (Table 3), the eigenvalues (relevant to those flow conditions) can be tentatively identified with a particular zone. This was done by seeding one zone, i, at a time and noting the λ_j associated with the dominant coefficient a_{ij}.

The eigenvalues were then compared with both the 'fresh air' infiltration rate and the 'total' air change rate into these zones. The total air change rate is evaluated by summing all the airflows, including the fresh air, flowing into that zone. Results are as shown in Table 4.

Table 4. Results.

Zone	Eigenvalue	Air change rate	Infiltration rate
1—rooms	3.28	4.98	3.13
2—corridor	41.3	31.9	0
3—roof void	12.2	20.0	0
4—stairwell	19.7	19.8	0
5—rest of building	0.839	1.16	0.787

This example shows that the eigenvalues do not necessarily equate either to the fresh air infiltration or to the total air change rate of a zone.

4.2 Effect of partially seeding a building

Fig. 2 shows concentration profiles in Zone 1 for the seeding patterns shown in Table 5.

Table 5. Concentration profiles.

Pattern of seeding	Percentage volume seeded
Zone 1	11
Zones 1, 2 & 4	16
All five zones	100

Fig. 3 shows the same profiles plotted on a semi-logarithmic (base 10) scale and, covering a range from 100 parts per million (ppm) to 1 part per billion (ppb), for an initial concentration level of 100 ppm. The upper range (100-1 ppm) can, with a suitable tracer, be measured using conventional infrared gas analysers[2,3] and the lower range (100-1 ppb) measured using gas chromatography[7].

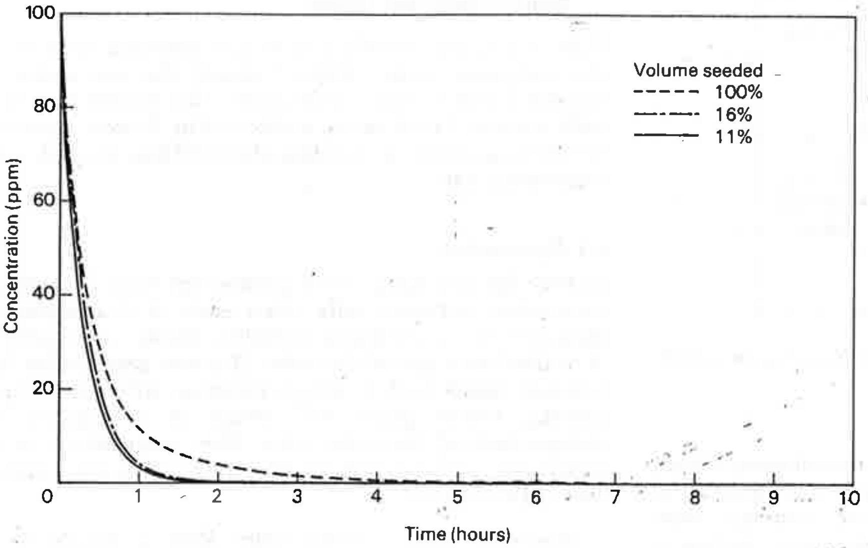

Fig. 2. Concentration profiles in Zone 1 for varying amount of 'seeded' volume.

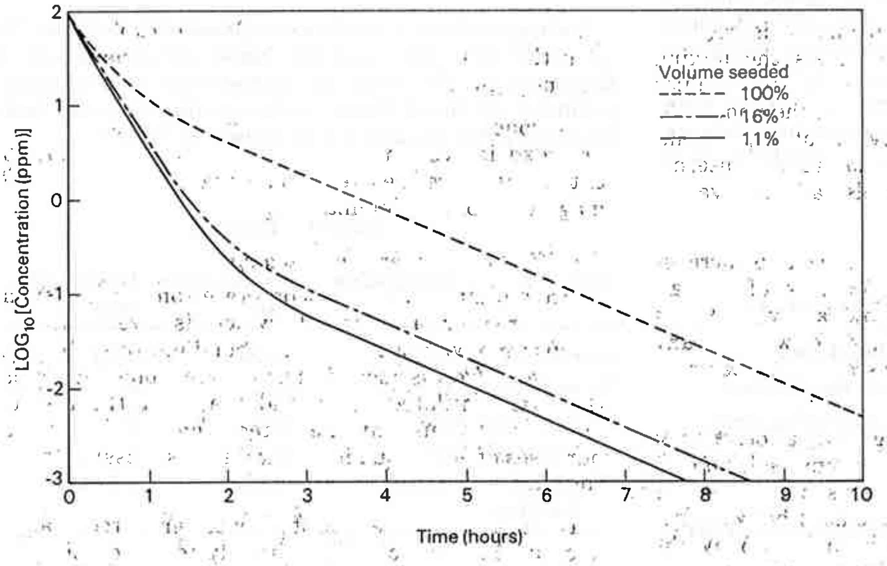

Fig. 3. Concentration profiles of Fig. 2 plotted logarithmically.

Fig. 3 shows that there are two distinct regions of decay. The initial part, called the 'transition' period[6], is made up from contributions from all the eigenvalues. A tracer gas 'decay' experiment will therefore show a continuously varying rate (as measured on a semilog plot) during this period. The consequence of this observation is that measurements carried out during the transition period do not necessarily reflect either the air change or infiltration rate of that zone or of the building.

An interesting point concerning the transition period arises when a seeding and decay experiment is carried out in a single zone. From Section 4.1, it is noted that the initial decay rate will closely approximate the eigenvalue 'associated' with that zone.

With increasing time, however, a 'dominant' decay mode is established. This period is almost entirely controlled by the smallest eigenvalue and the decay rate measured during this time is then equal to this dominant eigenvalue.

As more zones are seeded, the transition period gets shorter (Fig. 3) as the initial distribution is made a better approximation to the first eigenvector. The dominant period is then established earlier and the amount of tracer lost from that zone is reduced. The implication of this to practical measurements is twofold:

(a) the final dominant decay rate is independent of both the amount of tracer released or the locations of its release and

(b) the dominant decay rate can be obtained sooner and at a higher concentration level, for ease of measurement, by seeding as large a volume as possible.

4.3 Effect of enhancing interzone mixing

Theory states, and computer simulations show, that the eigenvalues are:

(a) independent of the initial tracer gas concentrations or distributions, but are

(b) dependent on the interzone airflows.

Interzone airflows can be enhanced by various means, e.g. by opening internal doors or using mixing fans. The effect of such enhanced mixing within the building were considered by carrying out computer simulations where an added 'mixing' flow β_3 (Fig. 1) was superimposed on the measured airflows.

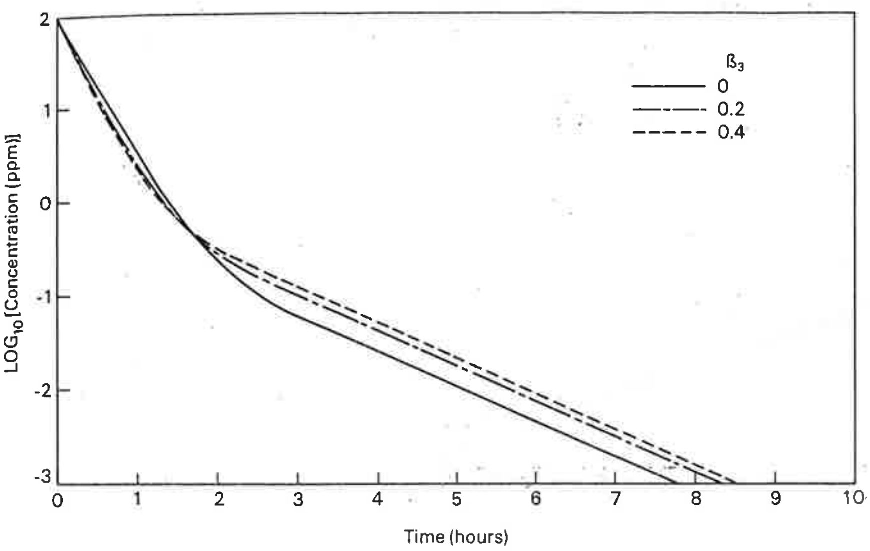

Fig. 4. Effect of enhanced mixing on Zone 1 concentration profiles.

The effect of enhancing the internal mixing is seen by considering Runs 1, 2 and 3 (Table 3). As the mixing increases, three out of the four relevant eigenvalues increase twofold in value. The smallest, and the dominant, eigenvalue however increases only by about 4 per cent. Other simulations carried out show that as the internal mixing increases above practical levels, all eigenvalues except the dominant become infinite.

The dominant eigenvalue, however, is seen to increase slowly towards the building infiltration rate of one air change per hour (using normalised airflows and volumes) as the mixing increases. It can be seen that, without any additional mixing, this eigenvalue was about 15 per cent away from the final value.

There is a physical reason why all eigenvalues other than the dominant increase to infinity. In any solution to Equation 1, one eigenvalue (which turns out to be the dominant) must represent the air movement between the building and the outside which has been set up by the external pressure field. If the mixing is balanced, i.e. there is no net flow in any direction, then there is relatively little disturbance upon the building envelope. The dominant eigenvalue therefore remains relatively unaffected with changes in intercell airflows.

The other eigenvalues, however, represent intercellular interaction. As the mixing gets better, the cells lose their individual identity and begin to communicate equally well with the outside. All eigenvalues, other than the dominant, then tend to infinity and the building approximates towards a single-cell structure.

A consequence of better internal mixing is that the transition period gets shorter and the 'dominant' decay is established earlier. The use of enhanced internal mixing (e.g. by using mixing fans) is therefore justified whenever whole building infiltration rates are required.

4.4 Effect of a stagnant zone

In most buildings, there are spaces (such as ceiling voids) which are not well-connected with the main ventilated space. To illustrate the possible effect of such a space, a simulation (Run 4 in Table 1) was carried out by connecting Zone 3 (ceiling void) to Zone 2 (corridor) with $\beta_2 = 0.1$.

The concentration profile generated in Zone 1 (rooms) from the resulting eigenvalues did not deviate to any discernible level from the profile (Figs. 2 and 3) when $\beta_2 = 0$ for the same flow distribution (Run 1). Table 1 also shows that the influence of the void on the dominant eigenvalue was negligible.

In conclusion, it is suggested that a stagnant zone, provided its volume is relatively small, does not to any extent influence the overall dispersion of the tracer nor the magnitude of the dominant eigenvalue.

4.5 Effect of an over-ventilated space

Office buildings usually incorporate some space, such as a foyer area or a stairwell, which is over-ventilated in comparison with the rest of the building. In such instances there are two distinct airchange rates, one characterising the over-ventilated space and the other the rest of the building. It is usually the latter value which is of use for purposes of either quality of air (i.e. freshness) or for energy considerations.

To determine the effect of an over-ventilated space, Zone 4 (stairwell) was connected directly to the outside with β_1 set to 0.2 (Run 5) and 0.4 (Run 6). This meant that up to 20 and 40 per cent of the total inflow of air into the building flowed into and out through Zone 4.

The resulting eigenvalues are given in Table 3. In both cases, the dominant eigenvalue has been reduced from 0.840 (Run 1) to 0.756 and 0.613. It can also be seen that they compare favourably with the infiltration rates of 0.827 and 0.620 respectively, into the rest of the building, i.e. excluding the over-ventilated Zone 4.

Fig. 5 shows concentration profiles in Zone 1 (with tracer seeding in Zone 1) for Runs 1, 5 and 6. They all show the steady 'dominant' decay rate emerging after the transition period. The time, however, at which the breakpoint between the transition and the dominant period occurs increases with an increase in the over-ventilation in Zone 4.

5 Conclusions

The major conclusions that may be drawn from this study are as follows:

(a) All concentration profiles will contain information regarding the eigenvalues characterising the building regardless of the position of tracer seeding or the extent to which the building is seeded.

(b) All profiles will consist of a transition period and a steady dominant period.

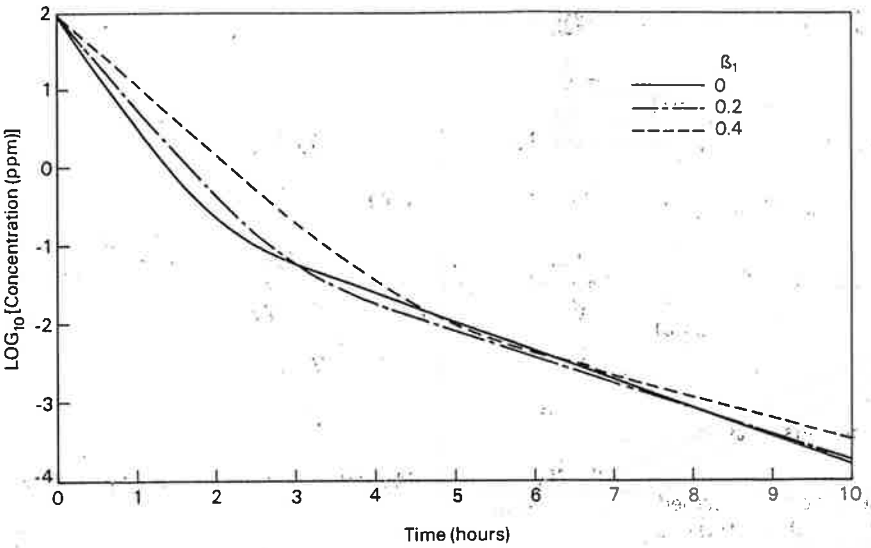

Fig. 5. Effect of an over-ventilated stairwell on Zone 1 concentrations.

(c) Decay rates obtained during the early transition period of a tracer 'decay' experiment do not necessarily give any meaningful information with regard to infiltration rates.
(d) After the transition period, the dominant period occurs during which the tracer decays at a constant rate. Monitoring in this dominant period may require a tracer gas capable of being detected at very low concentrations. For example, sulphur hexafluoride could be used and detected at ppb levels using an electron-capture detector coupled to a gas chromatograph.
(e) The transition region can be shortened and the dominant period reached quicker by
—increasing the volume seeded
—increasing the internal mixing by opening internal doors, using mixing fans, etc.
(f) There is always a measurable dominant decay rate which characterises a building. In a well-connected building, this approximates either to the total infiltration rate of the building, or at least to the infiltration rate of some significant portion of the building, e.g. the portion connected to an over-ventilated space.
(g) Measurements should be localised to the zone of seeding, thereby ensuring that tracer samples can be obtained relatively easily.

6 Future work

The approach presented here is, in principle, applicable to a wide variety of buildings whether they are small or large, single-celled or multicelled, naturally or mechanically ventilated. Field work to evaluate this technique will, however, concentrate on large, naturally ventilated office buildings. Measurements will incorporate procedures ranging from on-line to simple 'grab' sampling. If the results are successful it promises to lead to very economical methods for measuring ventilation rates with a reasonable degree of accuracy.

Acknowledgements

Thanks are due to Martin Shaw and John Burdett (Building Research Establishment) for writing the computer program used here and a special thank you to Mats Sandberg (National Swedish Institute for Building Research) for stimulating and thought-provoking discussions.

The work described is funded in part by the EC under Contract Number EE-A5-050-UK and has been carried out as part of the research programme of the Building Research Establishment of the Department of the Environment and this paper is published by permission of its Director.

References

[1] Perera, M. D. A. E. S., Review of techniques for measuring ventilation rates in multicelled buildings. Energy Conservation in Buildings—Heating Ventilation and Insulation (Ed. Ehringer, H., Hoyaux, G. and Zegers, P.), (Reidl Publishing Company, Dordrecht 1982).

[2] Perera, M. D. A. E. S., Walker, R. R. and Oglesby, O. D., Ventilation rates and intercell airflow rates in a naturally ventilated office building. Presented at the 4th Air Infiltration Centre Conference on Air Infiltration Reductions in Existing Buildings held in Elm, Switzerland on 26-28 September 1983.

[3] Perera, M. D. A. E. S., Walker, R. R. and Trim, M. J. B., Measurements of intercell airflows in large buildings using multiple tracer gases, Energy Saving in Buildings —Results of the European Communities Energy R & D Demonstration Programmes 1979-1983, (Ed. Ehringer, H. and Zito, U.). (Reidl Publishing Company, Dordrecht 1983).

[4] Lancaster, P., Theory of Matrices, (Academic Press, New York and London, 1969).

[5] Price, P. M., The BRE 'low energy' office building, BSER&T, 3 (3), 127-133 (1982).

[6] Hernandez, T. L. and Ring, J. W., Indoor radon source fluxes: experimental tests of a two-chamber model, Environment International, 8, 45-57 (1982).

[7] Breum, N. O. and Perera, M. D. A. E. S., Ventilation measurements using spot sampling of sulphur hexafluoride on a solid absorbent. Building and Environment, **19** (3), 175-178 (1984).

APPENDIX

Eigenvalues from a two cell model

Let the two cells 1 and 2 have volumes V_1 and V_2 and let the tracer concentration in each cell at time t be $C_1(t)$ and $C_2(t)$ respectively.

Let Q_{ij} be the airflow from cell i (= 0,1,2) to cell j (= 0,1,2) where the zeroth cell is the outside air.

If the outside air is at a zero tracer concentration and no tracer is produced within the cells once they have been initially seeded, then the conservation of the mass of tracer gives,

$$V_1 \dot{C}_1 = -(Q_{10} + Q_{12})\, C_1 + Q_{21}\, C_2 \quad (A.1)$$

$$V_2 \dot{C}_2 = Q_{12}\, C_1 - (Q_{10} + Q_{21})\, C_2 \quad (A.2)$$

where the independent variable t has been dropped for clarity.

The solutions to the above equation are given by,

$$C_1(t) = a_{11}e^{\lambda 1t} + a_{12}e^{\lambda 2t} \quad (A.3)$$

$$C_2(t) = a_{21}e^{\lambda 1t} + a_{22}e^{\lambda 2t} \quad (A.4)$$

where λ_1 and λ_2 are the eigenvalues and the constants a_{ij} (i, j = 1 or 2) account for initial conditions.

The eigenvalues are obtained from the system described by Equations A.1 and A.2 by solving the characteristic equation,

$$\left(-\frac{Q_{10} + Q_{12}}{V_1} - \lambda\right)\left(-\frac{Q_{20} + Q_{21}}{V_2} - \lambda\right) - \frac{Q_{21}\, Q_{12}}{V_1 V_2} = 0 \quad (A.5)$$

The eigenvalues are then given by,

$$-2\lambda = (r_1 + r_2) + (m_1 + m_2) \pm \sqrt{[\{(r_1 + m_1) - (r_2 + m_2)\}^2 + 4m_1m_2]} \quad (A.6)$$

where $r_1 = Q_{10}/V_1$, $r_2 = Q_{20}/V_2$,

and $m_1 = Q_{12}/V_1$, $m_2 = Q_{21}/V_2$

Equation A.6 can be reformulated by considering the equations governing the conservation of mass (neglecting small changes in density) of air,

$$r_1V_1 = Q_{01} + m_2V_2 - m_1V_1 \quad (A.7)$$

$$r_2V_2 = Q_{02} + m_1V_1 - m_2V_2 \quad (A.8)$$

Substituting these two equations in Equation A.6, we obtain the solution

$$-2\lambda = \left[\frac{Q_{01}}{V_1} + \frac{Q_{02}}{V_2}\right] + \left[\frac{m_2V_2}{V_1} + \frac{m_1V_1}{V_2}\right] \pm \sqrt{\left[\left\{\frac{Q_{01}}{V_1} - \frac{Q_{02}}{V_2} + \frac{m_2V_2}{V_1} - \frac{m_1V_1}{V_2}\right\}^2 + 4m_1m_2\right]} \quad (A.9)$$