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The Multi-chamber Theory Reconsidered from the Viewpoint of Air Quality Studies

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A general multi-chamber model is presented and explored from the viewpoint of air quality studies. The model involves the following key concepts: purging flow rate and age distribution of both air and contaminants. From the physical and mathematical properties of the model, are deduced estimates of the magnitude of, and the relations between, the key concepts. The practical use of the model is illustrated.

NOMENCLATURE

- A_{ij} cofactor of the flow matrix $\mathbf{Q}$
- b_{ij} elements of the inverse $\mathbf{Q}^{-1}$ of the flow matrix
- C concentration
- $\mathbf{C}$ column matrix (vector) whose elements are the chamber concentrations
- $C^{(\infty)}$ equilibrium concentration
- C_e extract concentration or equivalent extract concentration
- C_s supply concentration or equivalent supply concentration
- $\mathbf{e}$ eigenvector
- f statistical age frequency distribution of air or contaminant leaving the system
- $\mathbf{I}$ unit matrix
- m amount of contaminant or tracer gas released in a short burst
- $\dot{m}$ flow rate of contaminant or tracer gas
- $\partial\dot{m}$ flow rate of contaminant per chamber or room volume, $\partial\dot{m} \equiv \dot{m}/V$
- n nominal air exchange rate, $n \equiv Q/V$
- P_{pi} probability of a particle released at chamber i to pass into chamber p
- T transfer index
- Q total volumetric flow rate of air supplied to the system
- $\mathbf{Q}$ flow matrix
- $\mathbf{Q}^T$ transposed matrix obtained by interchanging the rows and columns of $\mathbf{Q}$
- U_p purging flow rate
- V total volume of system
- $\mathbf{V}$ volume matrix, whose entries are the chamber volumes
- X fraction of total supply air flow ($X \leqslant 1$)
- Y fraction of total extract air flow ($Y \leqslant 1$)
- $\mathbf{1}$ column matrix (vector) whose elements are unity
- $\mathbf{0}$ column matrix (vector) whose elements are zero
- β factor representing the state of mixing or secondary flow
- $\mathscr{H}$ fraction of total volume ($\mathscr{H} \leqslant 1$)
- $\mu^{(n)}$ nth moment about the origin of the age frequency distribution or concentration
- λ eigenvalues to the $\boldsymbol{\tau}$-matrix
- λ_e exponent of the concentration decay curve of exponential decay
- τ generic symbol for time or dummy variable in integration
- $\hat{\tau}$ non-dimensional time, $\hat{\tau} \equiv \tau/\tau_n$
- τ_n nominal time constant of ventilation system, $\tau_n = V/Q$
- $\boldsymbol{\tau}$ τ-matrix, $\boldsymbol{\tau} \equiv \mathbf{Q}^{-1}\mathbf{V}$
- τ' dummy variable in integration
- ϕ statistical internal age frequency distribution.

Subscripts and other symbols

- e refers to extract
- s refers to supply
- $\langle\ \rangle$ system-average
- $\sim$ refers to estimated quantity

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INTRODUCTION

THE TRANSPORTATION and mixing process in any flow system is, from the physical point of view, often conceptually divided into a 'systematic' part represented by the fluid velocity, and a 'random' part. The latter is due to molecular diffusion and turbulence. The molecular diffusion is represented by a molecular diffusion coefficient, and in practice one often tries to represent the action of turbulence by defining a turbulent mixing coefficient, analogous to the molecular diffusion coefficient. The mixing process can be represented by various models. Any model representing a real system can only serve as an approximation. One obvious criteria for selecting a certain model is physical relevance. Another criteria may be simplicity of model.

There are two main approaches to model turbulent flow systems. One approach is the use of the so-called advection (convection)–diffusion equation based on the concepts given above. The diffusion coefficients in the model are usually estimated by tracer experiments. The concentration of tracers are measured, and the diffusion coefficients are calculated by finding a solution to the advection–diffusion equation. This approach has been mainly used in unidirectional flow systems where two spatial dimensions are averaged out and the equation becomes a one-dimensional equation written in the direction of the main flow. The turbulent diffusion coefficient is replaced by a bulk dispersion coefficient, which considers both the effect of the averaging procedure and the action of turbulence. Rivers are examples of flow systems where this approach has been applied. In three-dimensional flow systems, e.g. in ventilated rooms, the meaning of turbulent diffusivities is not always clear.

The other model approach is the lumped-parameter 'chamber-model'. When the internal air-flows in a building are being studied, it is very common to represent the building as consisting of a number of interconnected perfect mixing chambers. The basic mixing unit, the instantaneously and uniformly mixed chamber, may consist of anything from part of a room to a whole set of rooms. The assumption that the internal mixing process is built up by a number of interconnected completely mixed chambers is very often a reasonable approximation. When it comes to the formulation of a multi-chamber model and to derive the terms that constitute the model, we must make it clear as to what kind of air-flow field and properties of the air-flow we want to derive. To illustrate the point, we may take the volumetric flow rate of air between mixing chambers. We must make it clear to ourselves whether we want to estimate the net flow rate of outdoor air between the chambers, or the total flow rate of air, including the internal secondary air circulation.

The secondary air-flow circulation may be generated by various heat sources (including man), by temperature differences on surfaces, or by entrainment of air into the jet stream from the supply air terminal. Normally the secondary flow is much greater than the supplied air-flow. As an example, we may consider an office room with 40 m^3 volume and a floor area of 15 m^2. The flow rate of outdoor air amounts to 150 m^3/h. If we assume that the supplied air-flow is uniformly distributed over an area equal to the floor area, we obtain an average throughput velocity equal to $150/(15 \times 3600) \approx 3 \times 10^{-3}$ m/s. However, we know that in this kind of situation the average velocity in the occupied chamber amounts to 0.10–0.20 m/s [1]. Therefore we can say, in a simplified manner, that the air-flow in a room is like a box filled with turbulence, with almost no net flow. It is unlike duct or channel flow where the net flow rate is easily identified. At any point within the room, we have air and contaminants of different ages, and from velocity records we cannot trace the net flow rate at which a contaminant is removed from the system.

Furthermore, we must consider in the model formulation, the differences between factors that govern transient phenomena (e.g. decay of contaminant or tracers), and factors that govern equilibrium concentration. The starting point in the model formulation is the total flow rates of air between the chambers. The total flow rates include both the secondary flows and the transportation by turbulent diffusion. The total flow rate from one chamber to another may be greater than the total flow rate of outdoor air supplied to the whole system. From the total flow rates between the chambers and the chamber volumes the following key concepts are deduced:

(i) The purging flow rate, the net rate by which a contaminant is 'flushed' out of the system. The purging flow rate is always less than or equal to, the total flow rate of outdoor air supplied to the whole system.
(ii) The mean-age of air or contaminant, the age is counted as the time elapsed since the entrance of air into the system or the release of contaminant.
(iii) The transition probability, which is the probability that a contaminant released in a region will pass into another region.

Equilibrium concentrations at a region surrounding a point source are controlled by the net flow rate of air flushing the region, i.e. the amount of air passing through, and not how fast the air is arriving at the region under discussion. On the other hand, transient phenomena, for example, the decay of a contaminant concentration are also dependent on how fast the air is arriving. The purging flow rates and mean-ages are of course to some extent interrelated. A high purging flow rate means of course that the air rather quickly arrives at the point in question.

The aims of this article are:

1. To present a multi-chamber model based on concepts pertinent to air quality studies. The concepts are the same as those given in the author's previous article in this journal [2]; therefore, the nomenclature used in this article is, with one exception, fully in accordance with that in the previous article.
2. To derive again some of the relations given in the previous article, now based on a multi-chamber model. Derivations based on multi-chamber models are easier to understand and will therefore better highlight the meaning of the different concepts.
3. To derive new estimates of the magnitude and the relationships between the key concepts given above. These estimates are obtained by starting from the mathematical properties of the model.
4. To illustrate, using real measurements, the necessity of making an appropriate model formulation.

Throughout the whole paper we assume that:

(i) contaminants are dynamically passive, i.e. they follow the air movements in the room. In the Appendix the differences between dynamically active and dynamically passive contaminants are discussed;
(ii) the release rate of contaminants is much smaller than the supply flow rate of air;
(iii) the contaminants are conserved, i.e. that which enters the system also leaves the system;
(iv) all physical quantities are assumed to be mean values in the sense of ensemble averages.

EXAMPLES OF WHEN THE MIXING-CHAMBER APPROACH IS VALID

It is well known that, in buildings, there are often greater differences in mixing between rooms than within rooms [3]. Even with connecting internal doors open, the internal walls act as partitions that somewhat obstruct the diffusion of air between rooms. Therefore, buildings can often be satisfactorily represented by a set of interconnected mixing-chambers, where each room constitutes a mixing-chamber. Due to the stack effect, especially in buildings with natural ventilation, there is often a net flow of air from the first to the second floor. This causes a mixing pattern with a pronounced difference between the first and second floor, and a two-chamber representation may be sufficient in this case.

Figure 1 shows a small test house equipped with a fan-powered supply and extract system. The house had two rooms of equal size (the volume of each room was approx. 50 m^3) with a door connecting the two rooms. Air at room temperature was admitted to room 1 and extracted from room 2. Tracer gas technique was used to monitor the local mean-age at two points, each in the centre of both rooms at a height of approx. 1.7 m above floor level. Measurements were carried out both with the connecting door open and closed. When the connecting door was closed, a narrow

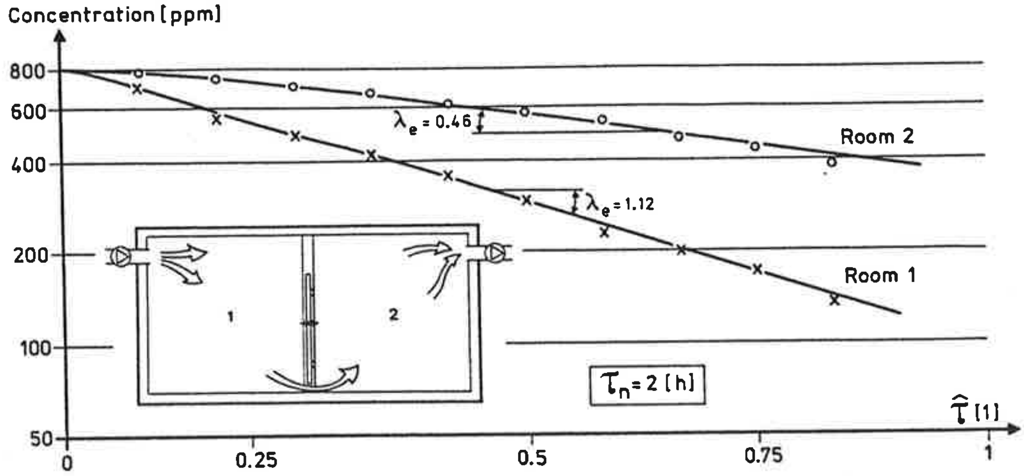

Fig. 1. Connecting door closed. Recorded tracer gas concentrations.

slot underneath the door allowed the air to pass from room 1 to room 2, but not in the opposite direction. The results are summarized in Table 1. When the connecting door was closed the one-way passage controlled the diffusion of the air in the whole house, and caused a considerable difference in mean-age between the rooms. That is we have two different mixing-chambers. Figure 1 shows the recorded tracer gas curves in a closed door case.

When the door was open the situation was somewhat different. At the lowest flow rate, the mean-age was about the same in both rooms and the whole house constituted one mixing-chamber. When the flow rate was increased, a pronounced two-chamber character appeared. The mixing in room 2 was however probably now not perfect at the higher flow rates. The mean-age of all air in the house was estimated by using relation (37c). A somewhat surprising difference appeared between the open door case and the closed door case. The mean-age of all air in the house was at its lowest when the door connecting the rooms was closed. The tracer gas (contaminant) was prevented from being spread back and forth throughout the house, and therefore was quickly removed from the house. This illustrates the fundamental objective of a ventilation system; that a contaminant shall be removed as quickly as possible.

Figure 2 shows the recorded tracer gas concentrations in a room to which both ventilation air and heat is supplied by a warm-air system (run A30 in [4]). Figure 3 shows the mean-age of air at several heights above floor levels. The two chamber character appears clearly both from the tracer gas curves and in the mean-age distribution.

SYSTEM AND MASS-BALANCE EQUATION

A room or a building, henceforth called a system, is subdivided into a arbitrary number, n, of chambers (see Fig. 4). In each chamber, the mixing is assumed to be uniform and instantaneous. The system boundary represents the boundary between outdoors and indoors. In general we have two types of chambers. The first type of chamber is directly linked to outdoors, outdoor air is either supplied directly to the chamber (supply air chamber), or air is transferred from the chamber to outdoors (extract air chamber). The second types of chambers are interior chambers which are only in contact with outdoors via other chambers.

It should be observed that when two indices appear, the first index denotes the desitination, while the second index denotes the origin. In the author's previous article [2], the order of the indices was in the reverse. The overall volumetric flow rate of air leaving the chamber is denoted by Q_{ii}.

$$Q_{ii} = \sum_{\substack{p=1 \\ (p \neq i)}}^{n} Q_{pi} + Q_{ei}. \tag{1}$$

The total volumetric flow rate of outdoor air to the whole system is:

$$Q = \sum_{i=1}^{n} Q_{is} = \sum_{j=1}^{n} Q_{ej}. \tag{2}$$

Furthermore we assume that in chamber i, a contaminant is released at a time-dependent rate $\dot{m}(\tau)$. Conservation of

Table 1. Recorded mean-age of air in test house in Fig. 1

		Connecting door open			Connecting door closed		
Nominal flow rate of air Q (m^3/h)	Nominal time constant τ_n (h)	Local mean-age of air Room 1 $\mu_{\Phi_1}^{(1)}$ (h)	Local mean-age of air Room 2 $\mu_{\Phi_2}^{(1)}$ (h)	Estimated mean-age of all air in the house $\langle \bar{\mu}_{\Phi}^{(1)} \rangle$ (h)	Local mean-age of air Room 1 $\mu_{\Phi_1}^{(1)}$ (h)	Local mean-age of air Room 2 $\mu_{\Phi_2}^{(1)}$ (h)	Estimated mean-age of all air in the house $\langle \bar{\mu}_{\Phi}^{(1)} \rangle$ (h)
25	4	4.00	4.10	4.05	1.91	4.53	3.22
50	2	1.39	2.70	2.04	0.95	2.26	1.61
100	1	0.80	1.37	1.08	0.60	1.08	0.84

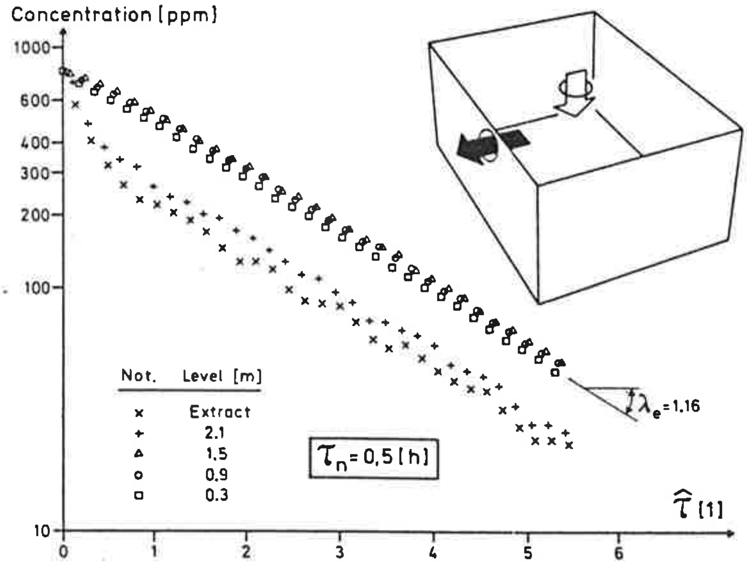

Figure 2. Short-circuiting system. Recorded tracer gas concentrations.

mass then gives for chamber i:

$$V_i\frac{dC_i}{d\tau} = -Q_{ii}C_{ii}+\sum_{\substack{j=1\\(i\neq j)}}^{n} Q_{ij}C_j+Q_{js}C_s+\dot{m}_i(\tau). \qquad (3a)$$

The equivalent exhaust concentration C_e, equation (17) in [2], becomes:

$$C_e = \frac{\Sigma Q_{ei}C_i}{Q}. \qquad (4a)$$

By using matrix notations equations, (3a) and (4a) can be written in compact form as:

$$\mathbf{V}\frac{d\mathbf{C}}{d\tau} = -\mathbf{QC}+\mathbf{Q}_s\mathbf{C}_s+\dot{\mathbf{m}}(\tau) \qquad (3b)$$

$$C_e = \frac{\mathbf{Q}_e^T\times\mathbf{C}}{Q} \qquad (4b)$$

where:

V = diagonal volume matrix with non-negative elements ($\geqslant 0$)

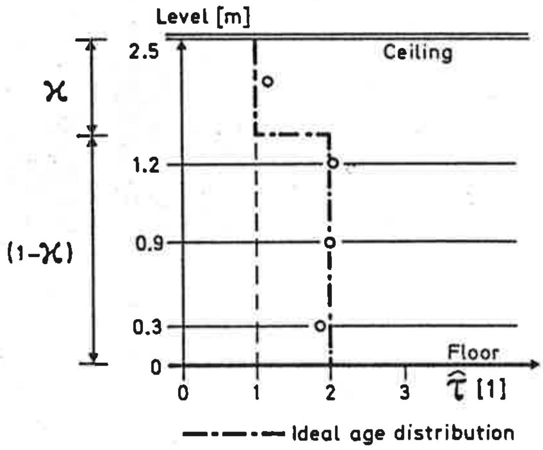

Fig. 3. Short-circuiting system. Recorded mean-age of air at several heights above floor level.

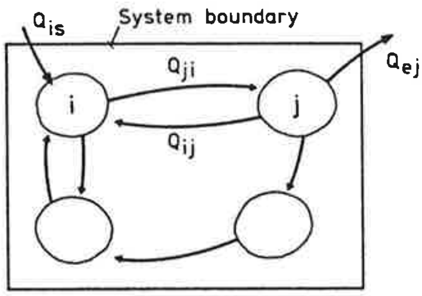

Fig. 4. Flow-system.

$\mathbf{C}_s$ and $\mathbf{C}_e$ are column matrices (vectors) with non-negative elements

Q is the quadratic flow matrix defined by:

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & -Q_{12} & \cdots & -Q_{1n} \\ -Q_{21} & Q_{22} & \cdots & -Q_{2n} \\ \vdots & \vdots & & \vdots \\ -Q_{n1} & -Q_{n2} & \cdots & Q_{nn} \end{bmatrix}.$$

The properties of the flow matrix and its inverse

A flow matrix has the following properties:

1. The flow matrix **Q** has positive diagonal elements and non-positive ($\leqslant 0$) off-diagonal elements. Furthermore we have the following constraints:

$$\mathbf{Q}_s = \mathbf{Q}\times\mathbf{1} \geqslant \mathbf{0}, \text{ i.e. } Q_{is} = Q_{ii}+\sum_{\substack{p=1\\(p\neq i)}}^{n}(-Q_{ip}) \geqslant 0, \qquad 1\leqslant i\leqslant n \quad (6a)$$

$$\mathbf{Q}_e = \mathbf{Q}^T\times\mathbf{1} \geqslant \mathbf{0}, \text{ i.e. } Q_{ej} = Q_{ii}+\sum_{\substack{p=1\\(p\neq i)}}^{n}(-Q_{pj}) \geqslant 0, \qquad 1\leqslant j\leqslant n \quad (6b)$$

where **1** is a column matrix whose elements are unity and **0** is a column matrix whose elements are zero.

Relation (6a) implies that the sum of all elements, for example in row i, are equal to the total flow rate of outdoor air supplied directly to chamber i. In an exactly analogous manner, relation (6b) implies that the sum of all elements in column j is equal to the total flow rate of air transferred directly from chamber j to outdoors. In the mathematical parlance, relation (6a) implies that the flow matrix $\mathbf{Q}$ is diagonally dominant, i.e.

$$Q_{ii} \geqslant \sum_{\substack{p=1 \\ (p \neq i)}}^{n} |-Q_{ip}| = \sum_{\substack{p=1 \\ (p \neq i)}}^{n} Q_{ip}. \tag{7}$$

Similarly, relation (6b) implies that the flow matrix is diagonally dominant with regard to the column sums. When the inequality (7) holds strictly, i.e. all row sums are greater than zero, then the matrix is said to be strict diagonally dominant.

The system is a closed system when there is no exchange of air between the system and outdoors. A naturally ventilated building can, when there is no wind or stack effect, occasionally be an almost closed system. Henceforth, we shall presuppose that there is always an exchange of air between the system and outdoors and therefore that the system is open, i.e.

$$\mathbf{Q}_s = \mathbf{Q}_e \neq \mathbf{0}. \tag{8}$$

This implies that at least one row sum, and one column sum, is greater than zero. We will in the sequel presuppose that the flow matrix is irreducible. This essentially means that there are no totally isolated chambers in the system. This is a very reasonable assumption; a chamber would at least be connected to another by molecular diffusion.

2. For an open system with an irreducible flow matrix, it is well known that the flow matrix, $\mathbf{Q}$, is non-singular and therefore its determinant is non-zero:

$$\text{Det } \mathbf{Q} \neq 0. \tag{9}$$

This implies that the inverse of the flow matrix, $\mathbf{Q}^{-1}$ exists. A non-singular flow matrix belongs to a sub-class of matrices called M-matrix ([5] Chap. 6). The inverse of the flow matrix has, as will be shown later, an important physical significance. The elements in the inverse matrix are given by:

$$\mathbf{Q}^{-1} = \frac{A_{ji}}{\text{Det } \mathbf{Q}} \equiv b_{ij} \tag{10}$$

where A_{ij} are the cofactors* of the matrix $\mathbf{Q}$.

3. The cofactors of the diagonal elements in $\mathbf{Q}$ are positive:

$$A_{ii} > 0 \tag{11a}$$

and the cofactors of the off-diagonal elements are non-negative [5]:

$$A_{ij} \geqslant 0 \quad (i \neq j). \tag{11b}$$

In view of relation (10), this shows that the elements in the inverse matrix are non-negative.

A theorem, derived by Ostrowski [6], for strict diagonally dominant matrices applied on the flow matrix gives that for the elements b_{ij} in $\mathbf{Q}^{-1}$ it holds:

$$\max_i b_{ij} = b_{jj} \quad \text{and} \quad \max_j b_{ij} = b_{ii} \tag{12}$$

i.e. the maximum element in a row or column of $\mathbf{Q}^{-1}$ is the diagonal element. Later, Berman and Plemmons [5], reported that (12) holds even when not all the row or column sums in $\mathbf{Q}$ are greater than zero.

The mass balance equation reformulated and its general solution

As we have seen, the inverse matrix $\mathbf{Q}^{-1}$ always exists and therefore we obtain after multiplying both sides in (3b) by $\mathbf{Q}^{-1}$:

$$\mathbf{Q}^{-1}\mathbf{V}\ d\mathbf{C}(\tau)/d\tau = -\mathbf{C}(\tau) + \mathbf{Q}^{-1}\mathbf{Q}_s\mathbf{C}_s + \mathbf{Q}^{-1}\dot{\mathbf{m}}(\tau). \tag{13a}$$

The physical dimension of the elements in the matrix $\mathbf{Q}^{-1}\mathbf{V}$ is time. The matrix $\mathbf{Q}^{-1}\mathbf{V}$ has, as will be shown further on, a particular physical interpretation, and therefore we name it the $\boldsymbol{\tau}$-matrix, that is:

$$\boldsymbol{\tau} \equiv \mathbf{Q}^{-1}\mathbf{V}. \tag{14}$$

Its inverse always exists and is equal to:

$$\boldsymbol{\tau}^{-1} = \mathbf{V}^{-1}\mathbf{Q}. \tag{15}$$

The physical dimension of the elements in the inverse matrix is the reciprocal of time.

After introducing the $\boldsymbol{\tau}$-matrix into (13a) it becomes:

$$\boldsymbol{\tau}\ d\mathbf{C}/d\tau = -\mathbf{C}(\tau) + \mathbf{Q}^{-1}\mathbf{Q}_s\mathbf{C}_s + \mathbf{Q}^{-1}\dot{\mathbf{m}}(\tau). \tag{13b}$$

Without any loss of generality we will in the sequel assume that the supply air concentration is equal to zero, that is:

$$\mathbf{C}_s = 0. \tag{16}$$

In case of no contaminant sources, $\dot{\mathbf{m}} = 0$. The time evolution of the concentrations is governed by:

$$\boldsymbol{\tau}\ d\mathbf{C}(\tau)/d\tau = -\mathbf{C}(\tau). \tag{17a}$$

By multiplying each side of equation (17a) by $\boldsymbol{\tau}^{-1}$, we obtain the mathematically equivalent expression:

$$d\mathbf{C}(\tau)/d\tau = -\boldsymbol{\tau}^{-1}\mathbf{C}(\tau). \tag{17b}$$

Starting from an initial concentration $\mathbf{C}(0)$ at time $\tau = 0$ the time history of the concentration is given by ([7], pp. 191–192):

$$\mathbf{C}(\tau) = (e^{-\boldsymbol{\tau}^{-1}\tau})\mathbf{C}(0) + \mathbf{V}^{-1} \times \left[\int_0^{\tau} e^{-\boldsymbol{\tau}^{-1}(\tau-\tau')}\dot{\mathbf{m}}(\tau')\,d\tau'\right]. \tag{18}$$

$e^{-\boldsymbol{\tau}^{-1}\tau}$ is the matrix exponential defined by a Taylor series ([7], p. 180):

$$e^{-\boldsymbol{\tau}^{-1}\tau} \equiv \sum_{n=1}^{\infty} \frac{1}{n!}(-\boldsymbol{\tau}^{-1}\tau) = \mathbf{I} - \boldsymbol{\tau}^{-1}\tau + \frac{1}{2!}\tau^2(\boldsymbol{\tau}^{-1})^2 + \cdots. \tag{19}$$

$\mathbf{I}$ is the unit matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{20}$$

*The cofactor A_{ij} of the element Q_{ij} in $\mathbf{Q}$ is defined as: $A_{ij} = (-1)^{i+j}D_{ij}$ where D_{ij} is the determinant of the matrix obtained by striking our row i and column j.

and $(\tau^{-1})^n$ is matrix multiplication of the matrix τ^{-1} by itself n times.

The first term in (18) is dependent on the initial concentration, while the second term is the 'source term'. As time increases, the first term decreases and becomes negligible compared to the 'source term'. With a constant flow rate of contaminant equation (18) becomes:

$$\mathbf{C}(\tau) = (e^{-\tau^{-1}\tau})\mathbf{C}(0) + \mathbf{Q}^{-1}\dot{\mathbf{m}}. \tag{21}$$

The equilibrium concentration attained is given by:

$$\mathbf{C}(\infty) = \mathbf{Q}^{-1}\dot{\mathbf{m}}. \tag{22}$$

In the case of no contaminant source (21) becomes:

$$\mathbf{C}(\tau) = (e^{-\tau^{-1}\tau})\mathbf{C}(0). \tag{23}$$

Equation (19) shows that with a time-dependent release rate of contaminant, the 'source term' contains terms dependent on both the flow rate of air and the volumes of the chambers. However, equation (21) shows that in the case of a constant rate of release of contaminant, the 'source term' is dependent only on flow terms.

In this subsection we have seen that the inverse, $\mathbf{Q}^{-1}$, of the flow matrix appears both in the expression for the equilibrium concentration attained (22), and in the expression for the time evolution of the concentrations (17a). This motivates a closer look at the elements in $\mathbf{Q}^{-1}$, and especially consideration of their physical meaning. However, before this can be done we need the moments with regard to time of the concentration histories. From the moments, we can deduce the age distributions of both the supplied air and contaminants released within the room.

The moments of concentration histories

The objective of this subsection is to derive a relation between the moments of concentration histories in interior chambers and the moments of concentration histories in the extract air chambers.

The moments about the origin are:

$$\mu_C^{(n)} = \int_0^\infty \tau^n C(\tau)\,d\tau \quad n = 0, 1, 2, 3, \ldots \tag{24}$$

where $\mu_C^{(n)}$ is the area under the curve.

After use of integration by parts, the application of definition (24) on relation (23) gives:

$$\mu_C^{(n)} = n!(\tau^{-1})^{-(n+1)}\mathbf{C}(0) = n!((\tau^{-1})^{-1})^{n+1}\mathbf{C}(0) = n!\tau^{(n+1)}\mathbf{C}(0). \tag{25}$$

aftering inserting (23) into the definition (4b) of the equivalent exhaust concentration we obtain:

$$C_e(\tau) = \frac{1}{Q}\mathbf{Q}_e^T[\exp(-\tau^{-1}\tau)]\mathbf{C}(0) \tag{26}$$

where, see equation (6b):

$$\mathbf{Q}_e^T = \mathbf{1}^T \times \mathbf{Q}. \tag{27}$$

We insert (27) into (26) and calculate the moments in accordance with (24) and obtain:

$$\begin{aligned}\mu_{C_e}^{(n)} &= \frac{n!}{Q}(\mathbf{1}^T \times \mathbf{Q} \times \tau^{n+1})\mathbf{C}(0)\\ &= \frac{n!}{Q}(\mathbf{1}^T \times \mathbf{V} \times \mathbf{V}^{-1}\mathbf{Q} \times \tau^{n+1})\mathbf{C}(0)\\ &= \frac{n!}{Q}(\mathbf{1}^T \times \mathbf{V}\tau^{-1}\tau^{n+1})\mathbf{C}(0)\\ &= \frac{n!}{Q}(\mathbf{1}^T\mathbf{V}\tau^n)\mathbf{C}(0)\end{aligned} \tag{28}$$

where $\tau^0 \equiv \mathbf{I}$.

From (25) we see that (28) can be expressed as:

$$\mu_{C_e}^{(n)} = \frac{n}{Q}(\mathbf{1}^T\mathbf{V}\mu_C^{(n-1)}). \tag{29}$$

We divide each side in (29) by the total volume V, and rearrange the terms, and obtain:

$$\frac{1}{V}(\mathbf{1}^T \times \mathbf{V})\mu_C^{(n-1)} = \frac{Q}{V}\frac{1}{n}\mu_{C_e}^{(n)}. \tag{30a}$$

The matrix multiplication in equation (30a) yields a summation of the moment, $\mu^{(n-1)}$ in each chamber weighted by the corresponding chamber's fraction of the total volume, that is:

$$\sum_{i=1}^{n}\left(\frac{V_i}{V}\right)\mu_{C_i}^{(n-1)} = \frac{Q}{V}\frac{1}{n}\mu_{C_e}^{(n)}. \tag{30b}$$

By definition, the left-hand side in (30b) is the system-average, $\langle\mu_C^{(n-1)}\rangle$, of the $(n-1)$th moment and therefore we rewrite (30b) as:

$$\langle\mu_C^{(n-1)}\rangle = \frac{Q}{V}\frac{1}{n}\mu_{C_e}^{(n)}. \tag{30c}$$

This is the same as relation (14) in [2].

The age distribution of the air

We possess different methods, based on tracer gas technique, for determining the mean-age, $\mu_{\phi_p}^{(1)}$, of the air in an arbitrary chamber p, see [2] Table 1. One possibility is to start from the same initial concentration in each chamber, that is:

$$\mathbf{C}(0) = C(0)\mathbf{1} \tag{31}$$

and record the concentration decay. The mean-age is obtained by dividing the 0th moment, $\mu_C^{(0)}$, i.e. the area under the decay curve by the initial concentration:

$$\mu_\phi^{(1)} = \frac{1}{C(0)}\mu_C^{(0)}. \tag{32}$$

According to (25):

$$\mu_C^{(0)} = \tau \times \mathbf{C}(0). \tag{33a}$$

Combining (31) and (33a) yields:

$$\mu_C^{(0)} = \tau \times \mathbf{1}C(0) \tag{33b}$$

Inserting (33b) into (32) gives the important relation:

$$\mu_\phi^{(1)} = \tau \times \mathbf{1}. \tag{34}$$

The right-hand side in (34) is a column matrix consisting of the row sums of matrix τ. That is, the sum of the elements in for example row i in the matrix τ is equal to the mean-age of the air in chamber i, $\mu_{\phi_i}^{(1)}$.

The mean-age of the air when it leaves the room, $\mu_f^{(1)}$, we obtain from the concentration readings in the extract duct as:

$$\mu_f^{(1)} = \frac{\mu_{C_e}^{(0)}}{C(0)}. \tag{35a}$$

According to (28) the 0th moment (area under the concentration curve) is:

$$\mu_{C_e}^{(0)} = \frac{1}{Q}(\mathbf{1}^T\mathbf{V}\mathbf{I})\mathbf{C}(0) = \frac{VC(0)}{Q} \tag{36}$$

and (35a) becomes:

$$\mu_f^{(1)} = \frac{V}{Q} \equiv \tau_n. \tag{35b}$$

Relation (35b), in the context of multi-chamber models, has been provided earlier by Wen and Fan [8], Chap. 7.

By inserting (4a) and (32) into (35a, b) the average age of all air leaving the system can be expressed in terms of the mean-age of the air in chambers from which the air is leaving the system:

$$\mu_f^{(1)} = \frac{\mu_{C_e}^{(0)}}{C(0)} = \sum_j \left(\frac{Q_{e_j}}{Q}\right)\mu_{\phi_j}^{(1)} = \tau_n. \tag{35c}$$

That is, the mean-ages are weighted by the flow rate of air from the chamber to outdoors, to the total flow rate of air from the system.

The mean-age of the air in the room, $\langle\mu_{\phi_p}^{(1)}\rangle$, is equal to:

$$\langle\mu_{\phi_p}^{(1)}\rangle = \frac{\mu_{C_e}^{(1)}}{\mu_{C_e}^{(0)}}. \tag{37a}$$

The first moment, $\mu_{C_e}^{(1)}$, according to (28) is equal to:

$$\mu_{C_e}^{(1)} = \frac{1}{Q}(\mathbf{1}^T\mathbf{V} \times \boldsymbol{\tau} \times \mathbf{1})C(0). \tag{38}$$

Using (36), (38) and (34) in (37a), we obtain the following expression for the mean-age of air in the whole system:

$$\langle\mu_{\phi}^{(1)}\rangle = \frac{(\mathbf{1}^T\mathbf{V} \times \boldsymbol{\tau} \times \mathbf{1})}{V} = \frac{\mathbf{1}^T\mathbf{V}\mu_{\phi_p}^{(1)}}{V}. \tag{37b}$$

The matrix multiplication in the denominator of (37b) gives rise to a summation of the mean-ages, $\mu_{\phi_i}^{(1)}$, in each chamber i, weighted by the corresponding chamber's volume fraction of the total volume, compare equation (67) in [2]:

$$\langle\mu_{\phi}^{(1)}\rangle = \sum_{i=1}^{n}\left(\frac{V_i}{V}\right)\mu_{\phi_i}^{(1)}. \tag{37c}$$

In reference [9] it was proved that if at each point a passive contaminant is released at a rate:

$$\partial\dot{m} = \frac{\dot{m}}{V} \tag{38a}$$

then the equilibrium concentration attained at each point p is fully controlled by the local mean-age of air at the point p:

$$C_p(\infty) = \partial\dot{m}\,\mu_{\phi_p}^{(1)}. \tag{39}$$

We shall derive (39) in the context of multi-zone models.

The contaminant generation vector $\dot{\mathbf{m}}$, now becomes:

$$\dot{\mathbf{m}} = \partial\dot{m}\,\mathbf{V}\mathbf{1}. \tag{40}$$

This inserted in equation (22) for the equilibrium concentrations gives:

$$\mathbf{C}(\infty) = \partial\dot{m}\,\mathbf{Q}^{-1}\mathbf{V}\mathbf{1}. \tag{41a}$$

In terms of the τ matrix that is equal to:

$$\mathbf{C}(\infty) = \partial\dot{m}\,\boldsymbol{\tau} \times \mathbf{1} \tag{41b}$$

which by (34) becomes:

$$\mathbf{C}(\infty) = \partial\dot{m}\,\mu_{\phi}^{(1)} \tag{41c}$$

which is the same as (39).

The age distribution of contaminants

The mean-age of a contaminant released within the system can be obtained by the same procedure as for air. The only difference is that now we start from an equilibrium concentration caused by a contaminant source. The equilibrium concentration is given by (22) and the concentration is not necessarily the same in each chamber.

The mean-age of the contaminant in chamber number i, $\mu_{\phi_i}^{(1)}$, is calculated from the concentration readings in that chamber as:

$$\mu_{\phi_i}^{(1)} = \frac{\mu_{C_i}^{(0)}}{C_i(0)}. \tag{42}$$

For later reference we rewrite (42) as:

$$\mu_{C_i}^{(0)} = C_i(0)\mu_{\phi_i}^{(1)}. \tag{43}$$

The mean-age of the whole mass of contaminant present in the system $\langle\mu_{\phi}^{(1)}\rangle$, is calculated from the concentration readings in the exhaust as:

$$\langle\mu_{\phi}^{(1)}\rangle = \frac{\mu_{C_e}^{(1)}}{\mu_{C_e}^{(0)}}. \tag{44a}$$

From (28) we obtain the following expression of the zero moment in the exhaust:

$$\mu_{C_e}^{(0)} = \frac{1}{Q}(\mathbf{1}^T\mathbf{V}\mathbf{C}(0)). \tag{45a}$$

By carrying out the matrix multiplication in (45a) we obtain:

$$\mu_{C_e}^{(0)} = \frac{V}{Q}\langle C(0)\rangle \tag{45b}$$

where $\langle C(0)\rangle$ is the average contaminant concentration in the system. By using (29) the first moment in the exhaust can be written as:

$$\mu_{C_e}^{(1)} = \frac{1}{Q}(\mathbf{1}^T\mathbf{V}\mu_C^{(0)}). \tag{46}$$

By using (45b) and (46) we obtain the following expression for the mean-age of the contaminant present in the system:

$$\langle\mu_{\phi}^{(1)}\rangle = \frac{(\mathbf{1}^T\mathbf{V}\mu_C^{(0)})}{V\langle C(0)\rangle}. \tag{44b}$$

By carrying out the matrix multiplication in the denominator of (44b) and using (43) we obtain the final expression:

$$\langle \mu_{\phi}^{(1)} \rangle = \sum_{i=1}^{n} \left(\frac{V_i \times C_i(0)}{V \langle C(0) \rangle} \right) \mu_{\phi_i}^{(1)}. \tag{44c}$$

That is, the system-average age is obtained by summing the mean-ages in each chamber weighted by the fraction of contaminant content in the corresponding chamber of the total contaminant content in the system.

The physical interpretation of the inverse matrix $\mathbf{Q}^{-1}$

According to (10), the inverse matrix $\mathbf{Q}^{-1}$ can be written in terms of the cofactors, A_{ij}, of the matrix $\mathbf{Q}$ as:

$$\mathbf{Q}^{-1} = \frac{1}{\operatorname{Det} Q} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \equiv \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}. \tag{47}$$

We already know from (11a) and (11b) that all elements in $\mathbf{Q}^{-1}$ are non-negative ($\geqslant 0$) and that the diagonal elements are always greater than zero.

A constant release of contaminant in chamber i gives rise, according to (22), to a contaminant concentration in chamber p equal to:

$$C_p = b_{pi} \dot{m}_i. \tag{48}$$

The physical implication of relation (12), which states that the maximum element in each row of $\mathbf{Q}^{-1}$ is the diagonal element, is that the maximum concentration occurs in the chamber where the contaminant is released.

By using (48), the elements in $\mathbf{Q}^{-1}$ can be written as:

$$b_{pi} = \frac{C_p}{\dot{m}_i}. \tag{49}$$

The right-hand side in (49) is 'the transfer index', T_{pi}, see [2] equations (58) and (59), between the injection chamber and chamber p. That is:

$$b_{pi} = T_{pi} \quad (p \neq i). \tag{50}$$

The reciprocal of the local purging flow rate in chamber i, U_i, is by definition, see [2] equation (71b), equal to:

$$\frac{1}{U_i} = \frac{C_i}{\dot{m}_i} = b_{ii} = \frac{A_{ii}}{\operatorname{Det} Q}. \tag{51}$$

The local purging flow rate, U_p, expresses the net rate at which a contaminant is transported out of the system.

Expressed in terms of the local purging flow rate, U_p, and the transfer index, T_{pi}, the inverse matrix becomes:

$$\mathbf{Q}^{-1} = \begin{bmatrix} \frac{1}{U_1} & T_{12} & \cdots & T_{1n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & \frac{1}{U_n} \end{bmatrix}. \tag{52a}$$

The transfer index can be written as, see [2] equations (72) and (75), as:

$$T_{pi} = \frac{P_{pi}}{U_p} \tag{53}$$

where P_{pi} is a non-dimensional (true) probability that gives the probability of a particle released in chamber i passing into chamber p. The flow rate of contaminant, $\dot{m}_p$, at an arbitrary point is given by:

$$\dot{m}_p = P_{pi} \dot{m}_i. \tag{54}$$

The equilibrium concentration at the same point may be expressed as:

$$C_p = \frac{\dot{m}_i P_{pi}}{U_p} = \frac{\dot{m}_p}{U_p}. \tag{55}$$

In terms of the local purging flow rate, U_p, and the probability, P_{pi}, the inverse matrix $\mathbf{Q}^{-1}$ becomes:

$$\mathbf{Q}^{-1} = \begin{bmatrix} \frac{1}{U_1} & \frac{P_{12}}{U_1} & \cdots & \frac{P_{1n}}{U_1} \\ \vdots & \vdots & & \vdots \\ \frac{P_{n1}}{U_n} & \frac{P_{n2}}{U_n} & \cdots & \frac{1}{U_n} \end{bmatrix}. \tag{52b}$$

Restrictions to the magnitude of the local purging flow rate

We shall make use of the determinant of the flow matrix $\mathbf{Q}$. By expansion of the $\mathbf{Q}$ by its pth row, see [7], p. 33, we obtain:

$$\operatorname{Det} \mathbf{Q} = -Q_{p1}A_{p1} - Q_{p2}A_{p2} \cdots + Q_{pp}A_{pp} \cdots - Q_{pn}A_{pn}. \tag{56}$$

Now from (51) we know that the local purging flow rate in chamber p is equal to:

$$U_p = \frac{\operatorname{Det} Q}{A_{pp}}. \tag{57}$$

Therefore by combining (6a), (56) and (57) we obtain:

$$U_p = Q_{ps} + \sum_{\substack{j=1 \\ (i \neq p)}}^{n} \left(1 - \frac{A_{pj}}{A_{pp}}\right) Q_{pj}. \tag{58}$$

We obtain a similar expression by expansion of the $\mathbf{Q}$ by its pth column and making use of (56) and (57):

$$U_p = Q_{ep} + \sum_{\substack{i=1 \\ (i \neq p)}}^{n} \left(1 - \frac{A_{ip}}{A_{pp}}\right) Q_{ip}. \tag{59}$$

All terms in (58) and (59), including the terms in brackets, are greater than or equal to zero. Therefore it follows, that the local purging flow rate is greater than or equal to the largest of Q_{ps} (flow rate of outdoor air supplied directly to chamber p) and Q_{ep} (flow rate of air transferred directly from chamber p to outdoors). That is we have the following lower bound on the local purging flow rate, U_p, in a chamber directly connected to outdoors:

$$U_p \geqslant \max(Q_{ps}, Q_{ep}) \equiv Q_p. \tag{60}$$

The lower bound, given by (60), is obvious if one considers the physical interpretation of the concept purging flow rate itself. For an interior chamber we cannot give a lower bound on the purging flow rate, without knowing the flow structure in detail. To obtain an upper bound we are going to make use of (6a), which after multiplication by $\mathbf{Q}^{-1}$ on both sides becomes:

$$\mathbf{Q}^{-1} \times \mathbf{Q}_s = \mathbf{Q}^{-1}\mathbf{Q} \times \mathbf{1} = \mathbf{I} \times \mathbf{1} = \mathbf{1}. \tag{61a}$$

By carrying out the matrix multiplication in (61a) we obtain the following set of equations:

$$b_{pp}Q_{ps}\sum_{\substack{j=1\\(j\neq p)}}^{n} b_{pj}C_{js} = 1 \quad (1 \leqslant p \leqslant n). \tag{61b}$$

This and (12) gives us the following estimate:

$$1 \leqslant b_{pp}\left(\sum_{j=1}^{n} Q_{js}\right) = b_p Q. \tag{62}$$

We know from (51) that $b_{pp} = 1/U_p$ and we obtain:

$$1 \leqslant \frac{Q}{U_p} \quad \text{or} \quad U_p \leqslant Q. \tag{63}$$

That is, in any chamber, the local purging flow rate is less than or equal to the total rate of outdoor supplied to the system. In particular, if the total flow rate of outdoor air is supplied directly to one chamber only, or all the air transferred from the system to outdoors is taken from one chamber only, then both the lower bound (60) and the upper bound (63) are attained simultaneously. Therefore in this particular case we find that the local purging flow rate in the chambers in discussion are:

$$U_p = Q. \tag{64}$$

The matrix τ and restrictions to the local mean-age of air

By carrying out the matrix multiplication in definition (14) of the matrix τ, using expression (52b) of the matrix $\mathbf{Q}^{-1}$, we obtain:

$$\tau \equiv \mathbf{Q}^{-1}\mathbf{V} = \begin{bmatrix} \frac{V_1}{U_1} & \frac{V_2}{U_1}P_{12} & \cdots & \frac{V_n}{U_1}P_{1n} \\ \vdots & \vdots & & \vdots \\ \frac{V_1}{U_n}P_{n1} & \frac{V_2}{U_n}P_{n2} & \cdots & \frac{V_n}{U_n}P_{nn} \end{bmatrix}. \tag{65}$$

We know from relation (34) that the sum of elements in an arbitrary row p in τ is equal to the mean-age, $\mu^{(1)}_{\Phi_p}$, in chamber p, that is

$$\mu^{(1)}_{\Phi_p} = \frac{1}{U_p}\sum_{j=1}^{n} V_j P_{pj} = \frac{V_p}{U_p} + \frac{1}{U_p}\sum_{\substack{j=1\\(j\neq p)}}^{n} V_j P_{pj}. \tag{66}$$

We know from (55) that $P_{pj} \leqslant 1$ and therefore we obtain the following upper bound of the local mean-age:

$$\mu^{(1)}_{\Phi_p} \leqslant \frac{1}{U_p}\sum_{j=1}^{n} V_j = \frac{V}{U_p} \tag{67a}$$

or

$$U_p \times \mu^{(1)}_{\Phi_p} \leqslant V, \tag{67b}$$

compare equation (86) in [2].

Relation (67b) connects two important quantities, the local purging flow rate and the local mean-age of air. We can rewrite (67b) as:

$$U_p \leqslant \frac{\tau_n}{\mu^{(1)}_{\Phi_p}} Q. \tag{67c}$$

When $\mu^{(1)}_{\Phi_p} > \tau_n$, relation (67c) gives rise to the following restriction to the local purging flow rate:

$$U_p < Q \quad (\text{when } \mu^{(1)}_{\Phi_p} > \tau_n). \tag{68}$$

The reader should observe that the important relations (67) and (68) can be deduced in two different ways. One can either use a pure mathematical argument or start from the physical interpretation of the elements constituting the matrix τ. The mathematical argument is based on relation (12) which states that the diagonal element is the largest element in each row or column of the matrix τ. The other argument is based on the fact that the probability P_{pj} must be less than or equal to 1. In [2], another argument was used to deduce (67b) and (68).

All elements of the right-hand side of (66) are greater than or equal to zero. Therefore we obtain the following lower bound on the local mean-age of the air in any chamber:

$$\mu^{(1)}_{\Phi_p} \geqslant \frac{V_p}{U_p}. \tag{69}$$

The term on the right-hand side is the ratio of the volume of the chamber to the net rate of the transmission through the chamber. It can therefore be interpreted as the mean residence time in the chamber, that is, the mean time period spent by a 'molecule' entering the chamber.

We know that it always holds that $U_p \leqslant Q$ and therefore we deduce from (69):

$$\mu^{(1)}_{\Phi_p} \geqslant \frac{V_p}{Q}. \tag{70}$$

By combining (67a) and (69) we obtain for an arbitrary chamber:

$$\frac{V_p}{U_p} \leqslant \mu^{(1)}_{\Phi_p} \leqslant \frac{V}{U_p}. \tag{71}$$

In particular, for a chamber directly connected to outdoors we obtain by using (60):

$$\frac{V_p}{U_p} \leqslant \mu^{(1)}_{\Phi_p} \leqslant \frac{V}{Q_p}. \tag{72}$$

Relation (35c) gives rise to a restriction on the mean-age in extract air chambers. In the case of several extract air chambers there are two possibilities: in each extract air chamber is $\mu^{(1)}_{\Phi_p} = \tau_n$, or is $\mu^{(1)}_{\Phi_p} < \tau_n$, while $\mu^{(1)}_{\Phi_p} > \tau_n$ in at least one chamber.

An estimate of the magnitude of the largest eigenvalue of the matrix τ

In the next subsection, we need an estimate of the maximal eigenvalue of the matrix τ. The matrix is non-negative and therefore we may apply the Perron–Frobenius theorem ([7], Chap. 9). The Perron–Frobenius theorem states that the maximum eigenvalue is less, than or equal to, the maximum row sum, and greater than, or equal to, the minimum row sum. The theorem applied on τ and in view of the physical interpretation (34) of the row sums of τ gives:

$$\min_p \mu^{(1)}_{\Phi_p} \leqslant \max \lambda \leqslant \max_p \mu^{(1)}_{\Phi_p}. \tag{73}$$

From the matrix theory, we also know that the eigenvalue of the inverse matrix τ^{-1} are equal to the reciprocal of the eigenvalues of the matrix τ.

An estimate of the magnitude of the slope of an exponential decay curve

The decay of concentration is given by equation (23):

$$\mathbf{C}(\tau) = (e^{-\tau^{-1}\tau})\mathbf{C}(0).$$

When the matrix τ has n different eigenvalues, λ_K, then (23) can be expressed in a more explicit form, as:

$$\mathbf{C}(\tau) = \sum_{k=1}^{n} C_K e^{-\tau/\lambda_K} \mathbf{e}_k \tag{74}$$

where C_k are constants depending on the critical conditions and $\mathbf{e}_k$ are eigenvectors.

After a sufficiently long period of time has passed (say $\tau > \tau_0$), the right-hand side of (74) is dominated by the component with the largest eigenvalue, λ_k, that is we have a pure exponential decay:

$$\mathbf{C}(\tau) \sim e^{-\tau/\max \lambda_k} \quad (\tau > \tau_0) \tag{75}$$

where $\max \lambda_k$ is the maximum eigenvalue of the matrix τ.

Therefore in a plot of concentration vs time, the concentration curves in different chambers become parallel for $\tau > \tau_0$, see Fig. 2, with equal slope λ_e and it holds that:

$$\lambda_e = \frac{1}{\lambda_{\max}}. \tag{76}$$

Therefore, by combining (73) and (76), we obtain in this particular case:

$$\min_p \mu^{(1)}_{\phi_p} \leqslant \frac{1}{\lambda_e} \leqslant \max_p \mu^{(1)}_{\phi_p}. \tag{77}$$

That is, the magnitude of the reciprocal of the slope of an exponential decay curve lies between the smallest mean-age of air occurring in any chamber and the largest mean-age of air occurring in any chamber. Another significance of the slope, when pure exponential decay occurs is [2], [10]:

$$\lambda_e = \frac{Q}{V}\frac{C_e(\tau)}{\langle C(0)\rangle} \quad (\tau > \tau_0). \tag{78}$$

That is, the slope is directly related to the ratio between the concentration in the extract chambers and the system average concentration.

A MODEL CASE

A simple model case will be studied in some detail to give insight into the meaning of the concepts introduced. The system is a two-room house, see Fig. 5. Between the rooms there is a doorway. In each room the mixing is assumed to be complete and instantaneous.

The flow rate of outdoor air supplied to room 1 is assumed to be greater than or equal to the flow rate of air extracted from room 1, that is $X > Y$. This gives rise to a net flow rate of air from room 1 to room 2, which amounts to $(X-Y)\times Q$. The secondary flow is expressed as βQ, where β may vary from no secondary flow between the rooms ($\beta = 0$), up to the mathematical limit ($\beta = \infty$). The latter limit gives rise to complete and instantaneous mixing in the whole system. In practice the lower bound for complete mixing is attained for β equal to 5–10. The actual value depends to some extent on the relative position between the supply and extract points.

Based on the relations derived earlier, we can estimate the magnitude of the local purging flow rate and the local mean-age without doing any calculations. Because $XQ \geqslant YQ$ we obtain from (60) that for room 1, the purging flow rate is:

$$U_1 \geqslant XQ. \tag{79}$$

$X \geqslant Y$ implies that $(1-Y) \geqslant (1-X)$ and we conclude from (60) that for room 2 it holds that:

$$U_2 \geqslant (1-Y)Q. \tag{80}$$

For the local mean-age in chamber 1, relations (70) and (72) give us the estimate:

$$\mathscr{H}\frac{V}{U_1} \leqslant \mu^{(1)}_{\phi_1} \leqslant \frac{\tau_n}{X}. \tag{81}$$

Relations (70) and (72) give for room two:

$$(1-\mathscr{H})\tau_n \leqslant \mu^{(1)}_{\phi_2} \leqslant \frac{\tau_n}{(1-Y)}. \tag{82}$$

The flow matrix of the system consisting of the total flow rates is readily set up (the arrows denote the corresponding row or column sum):

$$\mathbf{Q} = Q\begin{bmatrix} (X+\beta) & -\beta \\ -((X-Y)+\beta) & (1-Y)+\beta \end{bmatrix} \begin{matrix} \rightarrow XQ \\ \rightarrow (1-X)Q \end{matrix} \tag{83}$$
$$\begin{matrix} \downarrow & \qquad \downarrow \\ YQ & \qquad (1-Y)Q. \end{matrix}$$

The determinant is calculated as being:

$$\operatorname{Det} \mathbf{Q} = Q^2(X(1-Y)+\beta). \tag{84}$$

The inverse matrix $\mathbf{Q}^{-1}$ becomes:

$$\mathbf{Q}^{-1} = \frac{1}{Q}\begin{bmatrix} \dfrac{(1-Y)+\beta}{X(1-Y)+\beta} & \dfrac{\beta}{X(1-Y)+\beta} \\ \dfrac{(X-Y)+\beta}{X(1-Y)+\beta} & \dfrac{X+\beta}{X(1-Y)+\beta} \end{bmatrix}. \tag{85}$$

We know from (51) that the reciprocal of the diagonal elements in $\mathbf{Q}^{-1}$ are the purging flow rates U_1 and U_2. The off-diagonal elements are P_{12}/U_1 and P_{21}/U_2. We read off from (85):

$$\begin{aligned} U_1 &= Q\frac{X(1-Y)+\beta}{(1-Y)+\beta}; \quad P_{12} = \frac{\beta}{(1-Y)+\beta} \\ U_2 &= Q\frac{X(1-Y)+\beta}{X+\beta}; \quad P_{21} = \frac{(X-Y)+\beta}{X+\beta}. \end{aligned} \tag{86}$$

The matrix τ becomes:

$$\tau = \mathbf{Q}^{-1}\mathbf{V} = \tau_n$$

$$\begin{bmatrix} \mathscr{H}\dfrac{(1-Y)+\beta}{X(1-Y)+\beta} & (1-\mathscr{H})\dfrac{\beta}{X(1-Y)+\beta} \\ \mathscr{H}\dfrac{(X-Y)+\beta}{X(1-Y)+\beta} & (1-\mathscr{H})\dfrac{X+\beta}{X(1-Y)} \end{bmatrix}. \tag{87}$$

The row sums in (87) are the mean-age of air in room 1 and room 2 respectively:

$$\mu^{(1)}_{\phi_1} = \tau_n\frac{\mathscr{H}(1-Y)+\beta}{X(1-Y)+\beta} \tag{82a}$$

$$\mu^{(1)}_{\phi_2} = \tau_n\frac{X-\mathscr{H}Y+\beta}{X(1-Y)+\beta}. \tag{82b}$$

The average age of all air in the room we calculate from (37c):

$$\langle \mu_{\phi}^{(1)} \rangle = \tau_n \frac{\mathscr{H}(\mathscr{H}-Y)+X(1-\mathscr{H})+\beta}{X(1-Y)+\beta}. \qquad (88)$$

For some extreme values of the parameters X, Y and β, the resulting purging flow rates, transition probabilities and the mean-age of air are listed in Table 2. Complete mixing occurs when $\beta = \infty$ and no back-mixing occurs when $\beta = 0$. The displacement system occurs when there is no back-mixing and all outdoor air is supplied to room 1 only, and all air is extracted from room 2 only.

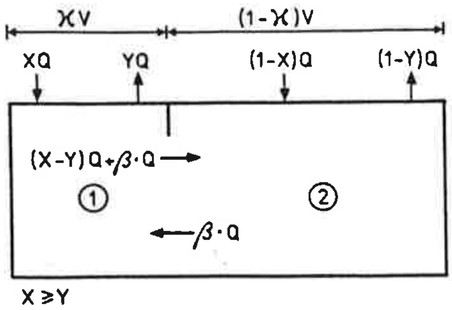

Fig. 5. Model system.

MEASUREMENTS

We possess different methods for determining the pertinent quantities, i.e. the purging flow rates and the age distributions of both air and contaminants. The following discussion will be limited to the determination of the purging flow rate and the age distribution of air. The methods can broadly be divided as follows:

1. *Pure experimental methods*

1a. *Direct method.* For each mixing chamber the quantities desired are monitored directly by the methods discussed in [2].

1b. *Determination of the flow matrix* **Q**. The entries in the flow matrix can for example be determined by multiple tracer gas technique. Then the inverse, $\mathbf{Q}^{-1}$, of the flow matrix gives us the desired quantities.

2. *Combined experimental technique and model approaches*

The age distribution of air is fairly easy to determine experimentally, while the purging flow rate is more difficult to determine. Therefore a useful approach may be to measure the age distribution and then, based on a model assumption, calculate the purging flow rate.

The application of methods 1b and 2 are illustrated below. The first example is based on measured interchamber flows and infiltration rates in a three-storey office building reported in reference [11]. Each floor was considered as a separate mixing chamber with a volume of 1762 m^3 each. The building had a mechanical ventilation system which allowed a variable amount of outdoor air to be taken into the building. This value could be varied from nominally zero (full recirculation) to full (no recirculation) outdoor intake. Figure 6 shows a schematic drawing of measured flow rates (m^3/s) from a test carried out with the ventilation system on full recirculation, i.e. nominally zero outdoor intake. The total amount of outdoor air entering the building amounted to 1.28 m^3/s (=4608 m^3/h). This corresponds to a nominal time constant equal to 1.15 (h). Beginning with the data from the second floor as the first row, the flow matrix **Q** becomes (entries in m^3/s):

$$\mathbf{Q} = \begin{bmatrix} 1.8 & -1.07 & -0.47 \\ -0.57 & 1.66 & -0.59 \\ -0.64 & -0.36 & 1.52 \end{bmatrix} \begin{matrix} \to 0.27 \\ \to 0.50 \\ \to 0.51 \end{matrix}.$$
$$\begin{matrix} \downarrow & \downarrow & \downarrow & \\ 0.60 & 0.22 & 0.46 & 1.28 \end{matrix}$$

The inverted matrix $\mathbf{Q}^{-1}$ becomes:

$$\mathbf{Q}^{-1} = \begin{bmatrix} 1.05 & 0.81 & 0.64 \\ 0.57 & 1.10 & 0.60 \\ 0.58 & 0.60 & 1.07 \end{bmatrix} \begin{matrix} \to 2.50 \\ \to 2.27. \\ \to 2.25 \end{matrix}$$

With the diagonal elements written as the reciprocal of the purging flow rates, and the off-diagonal elements written

Table 2. Model case, derived results

Type of system	Parameters X	Y	β	Purging flow rate Room 1 U_1	Room 2 U_2	Transition probability Room 2 → room 1 P_{12}	Room 1 → room 2 P_{21}	Local mean-age Room 1 $\mu_{\phi_1}^{(1)}$	Room 2 $\mu_{\phi_2}^{(1)}$
Complete mixing	0–1	$\leqslant X$	∞	Q	Q	1	1	τ_n	τ_n
No back-mixing	0–1	$\leqslant X$	0	XQ	$(1-Y)Q$	0	$\frac{X-Y}{X}$	$\frac{\mathscr{H}}{X}\tau_n$	$\frac{X-\mathscr{H}Y}{X(1-Y)}\tau_n$
Displacement flow	1	0	0	Q	Q	0	1	$\mathscr{H}\tau_n$	τ_n
All air supplied to chamber 1, all air extracted from chamber 2	1	0	–	Q	Q	$\frac{\beta}{1+\beta}$	1	$\frac{\mathscr{H}+\beta}{1+\beta}\tau_n$	τ_n
All air supplied to chamber 1	1	$\leqslant X$	–	Q	$\frac{(1-Y)+\beta}{1+\beta}Q$	$\frac{\beta}{(1-Y)+\beta}$	$\frac{(1-Y)+\beta}{1+\beta}$	$\frac{(1-Y)\mathscr{H}+\beta}{(1-Y)+\beta}\tau_n$	$\frac{1-\mathscr{H}Y+\beta}{(1-Y)+\beta}\tau_n$
All air extracted from chamber 2	0–1	0	–	$\frac{X+\beta}{1+\beta}Q$	Q	$\frac{\beta}{1+\beta}$	1	$\frac{\mathscr{H}+\beta}{X+\beta}\tau_n$	τ_n
Short circuiting	1	1	–	Q	$\frac{\beta}{1+\beta}Q$	1	$\frac{\beta}{1+\beta}$	τ_n	$\frac{1-\mathscr{H}+\beta}{\beta}\tau_n$

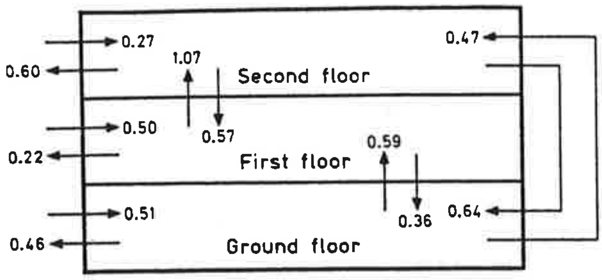

Fig. 6. Office building, inter-chamber air flows in m^3/s.

as transition probabilities divided by the purging flow rates, the inverse matrix $\mathbf{Q}^{-1}$ becomes:

$$\mathbf{Q}^{-1} = \begin{bmatrix} \frac{1}{0.95} & \frac{0.77}{0.95} & \frac{0.61}{0.95} \\ \frac{0.52}{0.91} & \frac{1}{0.91} & \frac{0.55}{0.91} \\ \frac{0.54}{0.93} & \frac{0.56}{0.93} & \frac{1}{0.93} \end{bmatrix}$$

Because each mixing chamber had the same volume, we obtain the row sums in the corresponding τ matrix by multiplying the row sums in the $\mathbf{Q}^{-1}$ matrix, by the volume of the mixing chambers. This gives us the mean-age of air on each floor. The results are summarized in Table 3.

Finally as an example of a mixed approach, we turn back to the example shown in Figs 2 and 3. It is a short-circuiting case. The volume fraction of the first mixing chamber, which all air is both supplied to and extracted from, we estimate to be $\mathscr{H} = 0.7$. From Fig. 3 we see that the local mean-age of air in chamber 2 is equal to $2\tau_n$. From Table 2, entry short-circuiting system, we read that in chamber 2 the local mean-age and the purging flow rate are:

$$\mu_{\phi_2}^{(1)} = \frac{1-\mathscr{H}+\beta}{\beta}\tau_n, \quad U_2 = \frac{\beta}{1+\beta}Q.$$

We know the local mean-age and this gives us $\beta = 0.7$, and subsequently the local purging flow rate becomes $U_2 = 0.41Q$.

CONCLUSIONS

The flow conditions occurring in ventilated spaces resemble those prevailing in recipients. The average air velocities caused by the total throughput flow is typically small compared to the velocities caused by secondary flows. Therefore at any point, air or contaminants of all ages are present, and a statistical approach is needed to quantify a system's performance to remove contaminants. In particular we must make a distinction between flow rates, predicted by recording velocities, and the net flow rates by which a contaminant is removed from the system.

The pertinent key concepts are the purging flow rates and the mean-age of both air and contaminants. A multi-chamber model based on these concepts has been formulated. Multi-chamber models are particularly suitable for representing the conditions in whole buildings, where each room is treated as a chamber. The starting point in the model formulation is the flow matrix $\mathbf{Q}$ consisting of the total flow rates of air between the chambers.

The total flow rates of air between chambers may well be greater than the total flow rate of outdoor air supplied. However, it turns out that the net flow rates (purging flow rates) are contained in the inverse $\mathbf{Q}^{-1}$ of the flow matrix. The reciprocal of the diagonal elements in $\mathbf{Q}^{-1}$ are the purging flow rates, while the off-diagonal elements are equal to a transition probability divided by a local purging flow rate. The inverse matrix $\mathbf{Q}^{-1}$ governs the equilibrium concentration attained, while the matrix $\tau = \mathbf{Q}^{-1}\mathbf{V}$ ($\mathbf{V}$ is the diagonal matrix of chamber volumes) governs the time evolution of concentration histories. The sum of the elements in a row, say row i, is equal to the local mean-age of air, $\mu_{\phi_i}^{(1)}$, in the corresponding chamber. The mass balance equation becomes:

$$\tau\frac{d\mathbf{C}}{d\tau} + \mathbf{C} = \mathbf{Q}^{-1}\dot{\mathbf{m}}.$$

Based on the mathematical properties of the matrix $\mathbf{Q}^{-1}$, it has been shown that the purging flow rate, U_{p_i}, is equal to or less than, the total flow rate, Q, of outdoor air.

This is fully consistent with the physical interpretation of the local purging flow rate. Furthermore, it has been shown, again starting from the mathematical properties of the matrix $\mathbf{Q}^{-1}$, that local purging flow rate and the mean-age of air an an arbitrary chamber are related as:

$$U_{p_i} \times \mu_{\phi_i}^{(1)} \leqslant V$$

where V is the total volume.

Based on this inequality it is readily seen that when the local mean-age of air is greater than (V/Q), then the flow is stagnant in the sense that the chamber is not 'flushed' by the total air flow supplied.

In case of exponential decay with the same and constant exponent λ_e, in each chamber, it has been shown that the

Table 3. Results obtained from the office building in Fig. 6

	Purging flow rate (m^3/h)	mean-age (h)	Transition probability							
Second floor	3420	1.22	Second $\uparrow$ First	0.77	Second $\uparrow$ Ground	0.61	—		—	
First floor	3276	1.10	First $\uparrow$ Ground	0.55	—		Second $\downarrow$ First	0.52	—	
Ground floor	3348	1.10	—		—		First $\downarrow$ Ground	0.54	Second $\downarrow$ Ground	0.56

exponent is related to the mean-age of air in the chambers:

$$\min \mu_{\phi_i}^{(1)} \leqslant \frac{1}{\lambda_e} \leqslant \max \mu_{\phi_i}^{(1)}.$$

Other estimates of the magnitude of the key concepts are reported in this article.

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APPENDIX

Dynamically active and dynamically passive contaminants

Both for a dynamically active and a dynamically passive contaminant, the equilibrium concentrations obtained at an arbitrary point p can be expressed as

$$C_p = \frac{\dot{m}_p}{U_p} = \frac{P_{pi}\dot{m}_i}{U_p}. \quad \text{(A1)}$$

The difference between a dynamically active and a dynamically passive contaminant appears in the transition probability, P_{pi}. Due to, for example, a high supply velocity or density difference with regard to the ambient air, a dynamically active contaminant sets up its own motion independent of local air motions. Therefore, the transition probability for an active contaminant is not necessarily directly related to the air motions, and subsequently, neither is it directly related to the flow matrix **Q**, used in the model approach in this article. For a passive contaminant however, the transition probability is governed by the inverse $\mathbf{Q}^{-1}$ of the flow matrix. The restriction (12) on the magnitude of the elements in an arbitrary row of $\mathbf{Q}^{-1}$ gives:

$$P_{pi} \leqslant 1. \quad \text{(A2)}$$

In accordance with the statistical interpretation of P_{pi}, this relation of course holds for both active and passive contaminants. The restriction (12) on the magnitude of the elements in an arbitrary column of the matrix $\mathbf{Q}^{-1}$ gives rise to the following relation:

$$P_{pi} \leqslant \frac{U_p}{U_{pi}}. \quad \text{(A3)}$$

In the case $U_{pi} < U_{pi}$, the restriction (A3) is stronger than the restriction (A2). By inserting (A3) into (A1), it follows that the maximum concentration is attained where the contaminant is released. When

$$P_{pi} = \frac{U_p}{U_{pi}} \leqslant 1 \quad \text{(A4)}$$

then it follows from (A1) that the equilibrium concentration obtained at an arbitrary point is the same as the concentration at the point of release.

This holds when, for example, a passive contaminant is released in the supply air duct. Then $P_{pi} = U_p/Q$ that is equal to the fraction of the total flow rate of air supplied that passes through the chamber (point p).

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