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Summary When it is required to predict the local mean age of air, the calculations are often carried out assuming that there is no recirculation, that is that all of the supply air is fresh. However, local mean age values will be greater than in the fresh air case when some of the exhaust air is recirculated. This paper examines and discusses the effect of recirculation and shows that, for many systems, the increase in local mean age due to recirculation can be calculated by means of an elementary analysis that leads to a simple algebraic expression. The paper also shows how the zonal model may be adapted for the general case of multiple recirculation paths. Compatibility between the elementary approach and the general zonal method is demonstrated.

Effect of recirculation on the local mean age of air

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List of symbols

$\bar{\tau}_e$	Mean age of exhaust air
$\bar{\tau}^r_e$	Mean age of exhaust air, with recirculation
$\bar{\tau}_{ei}$	Mean age of air at exhaust i
$\bar{\tau}_p$	Local mean age of air at point p
$\bar{\tau}^r_p$	Local mean age of air at point p, with recirculation
τ_r	Residence time
f_{ij}	Flow rate from zone i to zone j
r	Fraction of exhaust air recirculated
v_i	Volume of zone i
Q	Total ventilation air flow rate
Q_i	Air flow rate at point i
$\underline{\tau}$	local mean age vector
$\boldsymbol{F}$	multi-zone model flow matrix
$\boldsymbol{V}$	multi-zone model volume matrix
$\underline{u}$	-1 vector

1 Introduction

The local mean age of the air is now a well-known and useful measure of the distribution of fresh air in a ventilated space. However, for calculating values of the local mean age throughout a room, it is usually assumed that the air entering at the supply diffusers is wholly fresh. Although this is sometimes true, especially for natural ventilation, mechanical systems often recirculate a high proportion of the exhaust air. The supply air is therefore a mixture of fresh and recirculated air, and so values of the local mean age calculated on the basis of full fresh air are not correct. This problem was solved in a simplified way by Waters *et al.*[1] in a case study of a large open-plan office building. This paper reviews that solution and extends it to more general cases.

2 The simple case

The most basic type of ventilation system with a recirculation loop is shown in Figure 1. For a system such as this, the local mean age of air at the exhaust is in fact the residence time of the air leaving the chamber. It is shown by Etheridge and Sandberg[2] that the residence time, τ_r, of the air leaving the chamber, or external population, is independent of the degree of mixing within the chamber and is equal to V/Q and so it follows that

$$\bar{\tau}_e = \tau_r = \frac{V}{Q} \tag{1}$$

If a fraction r of the supply air is recirculated, the external population becomes $(Q - rQ)$ and so the age at the exhaust with recirculation, $\bar{\tau}^r_e$, becomes

$$\bar{\tau}^r_e = \frac{V}{Q - rQ} = \frac{\bar{\tau}_e}{1 - r} \tag{2}$$

The recirculated fraction, r, mixes with fresh air to create a mean age at the supply point $\bar{\tau}^r_s$, which is given by the weighted average of the ages of the two air streams:

$$\bar{\tau}^r_s = (1 - r)0 + \frac{r\bar{\tau}_e}{1 - r} = \frac{r\bar{\tau}_e}{1 - r} \tag{3}$$

If the mean age of the air at the supply point is known, this represents an additional term to the local mean age in the absence of recirculation, so that the mean age at a point p in the presence of recirculation, $\bar{\tau}^r_p$ is

$$\bar{\tau}^r_p = \bar{\tau}_p + \frac{r\bar{\tau}_e}{1 - r} \tag{4}$$

This provides a rapid method of calculating the effect of recirculation.

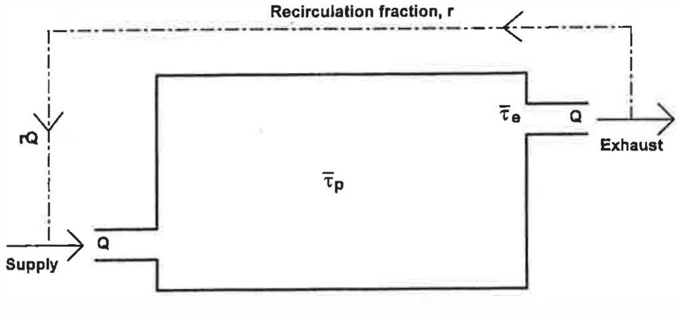

Figure 1 Simple system with recirculation

3 Extension to multiple exhausts

In practice, most systems operate with multiple supply diffusers and extracts. If the recirculated air passes through a single central air-handling unit, then all the supply diffusers receive and deliver air of identical quality, and so the correction to the local mean age will be the same as for the single diffuser case described above. However, in the case of multiple extracts, the air leaving each extract will, in general, have a different local mean age from the others. For the multiple exhaust example shown in Figure 2, the local mean ages at the exhausts in the absence of recirculation are $\bar{\tau}_{e,1}$, $\bar{\tau}_{e,2}$ and $\bar{\tau}_{e,3}$.

The total air flow rate through the space is again Q, and the outflows at the exhausts are Q_1, Q_2 and Q_3. In this case, with a fully mixing air-handling unit, the fraction recirculated from each extract will each be equal to the recirculation fraction applied to the supply air, r, and it is possible to apply the theory described above for the single supply and exhaust case.

The mean age of all of the air leaving the chamber, $\bar{\tau}_e$, i.e. the mean residence time, will be the weighted average of the ages at each of the three exhausts, $\bar{\tau}_{e,1}$, $\bar{\tau}_{e,2}$ and $\bar{\tau}_{e,3}$:

$$\bar{\tau}_e = \frac{\bar{\tau}_{e1}Q_1 + \bar{\tau}_{e2}Q_2 + \bar{\tau}_{e3}Q_3}{Q} = \frac{1}{Q}\sum_{i=1}^{3}\bar{\tau}_{ei}Q_i \tag{5}$$

where $Q = Q_1 + Q_2 + Q_3$.

Substituting for $\bar{\tau}_e$ in the equation for the single exhaust, it will be seen that for the multiple exhaust system shown in Figure 2, the exhaust local mean age is

$$\bar{\tau}_e^r = \frac{1}{(1-r)Q}\sum_{i=1}^{3}\bar{\tau}_{ei}Q_i \tag{6}$$

and so the local mean age at the supply point will be

$$\bar{\tau}_s^r = \frac{r}{(1-r)Q}\sum_{i=1}^{3}\bar{\tau}_{ei}Q_i \tag{7}$$

It follows that in the presence of recirculation the corrected local mean age for any point p is given by

$$\bar{\tau}_p^r = \bar{\tau}_p + \frac{r}{(1-r)Q}\sum_{i=1}^{3}\bar{\tau}_{ei}Q_i \tag{8}$$

A more general model than that described above is shown in Figure 3. In this case the recirculated component is returned independently from each of the three exhausts and so it is

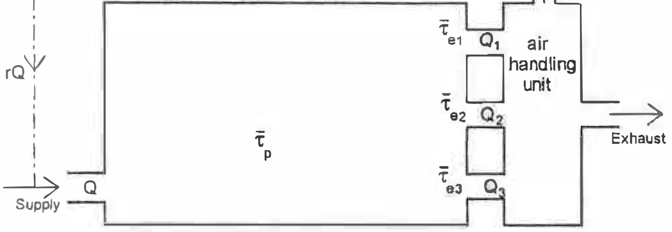

Figure 2 System with multiple exhausts, each having a common recirculation fraction

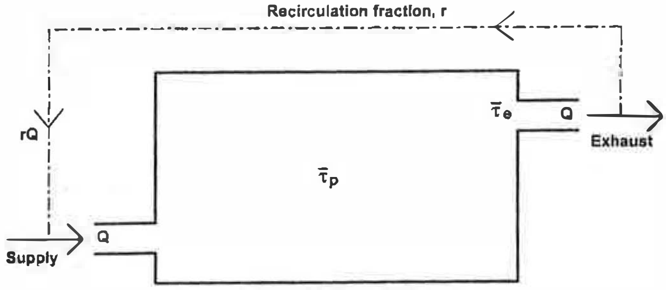

Figure 3 System with multiple exhausts, each with independent recirculation fractions

likely that the fractions r_1, r_2 and r_3 are not equal. However, from Figure 3 the overall recirculation fraction can be seen to be $rQ = r_1Q_1 + r_2Q_2 + r_3Q_3$.

The exhaust local mean age is

$$\bar{\tau}_e^r = \frac{1}{(1-r)Q}\sum_{i=1}^{3}\bar{\tau}_{ei}Q_i \tag{9}$$

which is the same as the case shown in Figure 2

For calculating the local mean age at the supply point, it is convenient to rewrite the equation for the case shown in Figure 2 as

$$\bar{\tau}_s^r = \frac{1}{(1-r)Q}\sum_{i=1}^{3}r\bar{\tau}_{ei}Q_i \tag{10}$$

However, for the case shown in Figure 3, r is not common to each of the three exhausts and so the equation must be written as

$$\bar{\tau}_s^r = \frac{1}{(1-r)Q}\sum_{i=1}^{3}r_i\bar{\tau}_{ei}Q_i \tag{11}$$

The local mean age at any point p may then be obtained from

$$\bar{\tau}_p^r = \bar{\tau}_p + \frac{1}{(1-r)Q}\sum_{i=1}^{3}r_i\bar{\tau}_{ei}Q_i \tag{12}$$

4 Zonal model

The above methods of allowing for recirculation are convenient if the local mean age for full fresh air is already known. Alternatively, recirculation can be included in the calculation of local mean age by use of a zonal model, with the advantage that such a model can easily accommodate any recirculation path. The simplest case, with a single supply and exhaust, is illustrated in Figure 4, in which the space is divided into 12 zones, with the supply feeding zone 9 and the exhaust extracting from zone 4. If f_{ij} represents the flow from zone i to zone j, the effect of the recirculation loop is to add an extra flow between the relevant zones (f_{49} in the case of Figure 4). This extra flow is in addition to any internal flow between the zones, although in practice the supply and exhaust zones will usually be sufficiently remote from each other to have no internal flow connection. The usual equation(3) for computing the local mean age for a zonal model is

$$\underline{\tau} = \boldsymbol{F}^{1}\boldsymbol{V}\underline{\boldsymbol{u}} \tag{13}$$

where $\underline{\tau}$ is the vector of local mean ages. The flow matrix is defined by

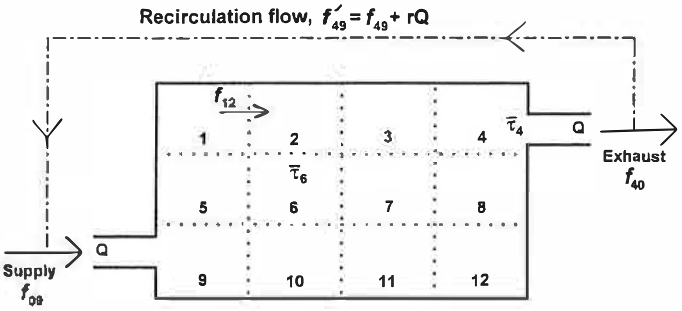

Figure 4 Simple zonal model

$$\boldsymbol{F} = \begin{bmatrix} -s_1 & -f_{21} & f_{31} & \cdots & f_{n1} \\ f_{12} & -s_2 & f_{32} & \cdots & f_{n2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & f_{3n} & \cdots & -s_n \end{bmatrix} \tag{14}$$

where f_{ij} = the flow from i to j.

$\boldsymbol{V}$ is an n-by-n diagonal matrix containing zone volumes and $\underline{\boldsymbol{u}}$ is a -1 vector as defined in equation 15.

$$\boldsymbol{V} = \begin{bmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_n \end{bmatrix} \qquad \underline{\boldsymbol{u}} = \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix} \tag{15}$$

Note that the flows to and from the outside air (denoted as the zero zone) do not appear in the flow matrix, and that the s_i entries on the diagonal are the sums of the total flows into and out of each zone. Flow rates to and from the external air are implied by the algebraic sum of each s_i and the flows in its corresponding row or column. Row sums give supply rates and column sums give exhaust rates. The effect of the recirculation loop, therefore, is to alter only the entry for the flow connecting the exhaust zone to the supply zone. The corresponding alterations to supply and exhaust (f_{09} and f_{40} in Figure 4) are taken care of by the row and column sums. Thus the recirculation from zone 4 to zone 9 in Figure 4 alters f_{49} to $f_{49}+rQ$; the rest of the flow matrix is unchanged.

In the case of recirculation from several exhaust zones to a common supply zone, there is an alteration in the flow matrix for each recirculation path. Figure 5, which is the zonal equivalent of Figure 3, gives rise to the following flow matrix:

$$\boldsymbol{F} = \begin{bmatrix} -s_1 & f_{21} & f_{31} & f_{41} & \cdots & f_{81} & \cdots & f_{12,1} \\ f_{12} & -s_2 & f_{32} & f_{42} & \cdots & f_{82} & \cdots & f_{12,2} \\ f_{13} & f_{23} & -s_3 & f_{43} & \cdots & f_{83} & \cdots & f_{12,3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{19} & f_{29} & f_{39} & f_{49}+r_4Q_4 & \cdots & f_{89}+r_8Q_8 & \cdots & f_{12,9}+r_{12}Q_{12} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{1,12} & f_{2,12} & f_{3,12} & f_{4,12} & \cdots & f_{8,12} & \cdots & -s_{12} \end{bmatrix} \tag{16}$$

It should be noted that $\boldsymbol{F}$ is a very sparse matrix because apart from s_i and r_iQ_i all entries are zero with the exception of elements f_{ij} where there is a direct flow connection from zone i to zone j. In the example shown in Figure 5, the flows f_{i9} in the terms $f_{i9} + r_iQ_i$ with i=4, 8 and 12 would all be zero, but

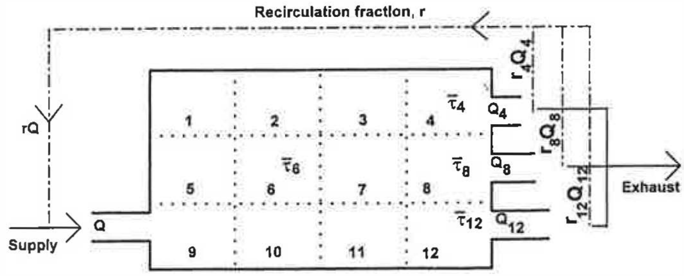

Figure 5 Zonal system with multiple exhausts

they are included here in matrix $\boldsymbol{F}$ in the interests of generality. Note also that the modified f_{i9} entries all appear on the ninth row. This is because their effect is to change the fresh air flow, f_{09}, into zone 9, and this is determined by the zone 9 row sum, s_9. It can easily be seen, therefore, that when the recirculated air is returned to more than one supply diffuser, that is to more than one zone, there will be a term $f_{ij}+r_iQ_i$ in every row that corresponds to a supply diffuser. In other words, entries will appear in the jth row and the ith column. This is illustrated in Figure 6, which is the same as Figure 5 but with two supply diffusers, so that entries now occur in rows 1 and 9, and in columns 4, 8 and 12.

Assuming that the recirculated air is divided equally between the two supplies, the flow matrix becomes

$$\boldsymbol{F} = \begin{bmatrix} -s_1 & f_{21} & f_{31} & f_{41}+\frac{1}{2}r_4Q_4 & \cdots & f_{81}+\frac{1}{2}r_8Q_8 & \cdots & f_{12,1}+\frac{1}{2}r_{12}Q_{12} \\ f_{12} & -s_2 & f_{32} & f_{42} & \cdots & f_{82} & \cdots & f_{12,2} \\ f_{13} & f_{23} & -s_3 & f_{43} & \cdots & f_{83} & \cdots & f_{12,3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{19} & f_{29} & f_{39} & f_{49}+\frac{1}{2}r_4Q_4 & \cdots & f_{89}+\frac{1}{2}r_8Q_8 & \cdots & f_{12,9}+\frac{1}{2}r_{12}Q_{12} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{1,12} & f_{2,12} & f_{3,12} & f_{4,12} & \cdots & f_{8,12} & \cdots & -s_{12} \end{bmatrix} \tag{17}$$

This makes explicit that exhaust zones are associated with columns and supply zones are associated with rows. Clearly, Figure 6 and its flow matrix can be generalised and adapted to suit any desired configuration.

5 Comparison between direct and zonal models

The direct model described in sections 2 and 3 should be consistent with the zonal models described in sections 3 and 4. Although consistency may be demonstrated by means of worked examples, a more rigorous proof of consistency is available. In the zonal model, the effect of recirculation may be considered as a perturbation to the flow matrix. However, the calculation of local mean age requires the inverse of the

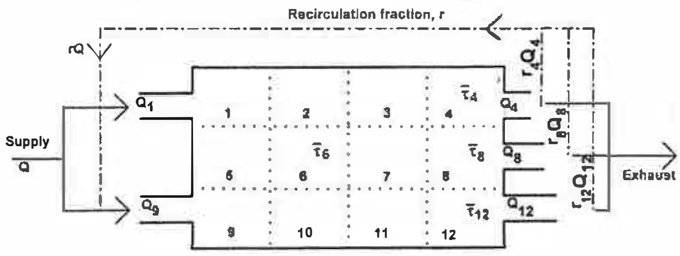

Figure 6 Zonal system with twin supplies and multiple exhausts

flow matrix. The effect of a matrix perturbation on the inverse of that matrix can be found by means of the Sherman–Morrison–Woodbury formula (see Golub and Van Loan[4]), which may be written as

$$(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}^{\mathrm{T}})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{I} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}^{-1} \tag{18}$$

where in general $\boldsymbol{A}$ is an n-by-n matrix, $\boldsymbol{B}$ and $\boldsymbol{C}$ are n-by-k matrices, and $\boldsymbol{I}$ is the identity matrix. The formula may be applied to the present problem by replacing $\boldsymbol{A}$ with the original flow matrix $\boldsymbol{F}$, that is without recirculation, and setting $k = 1$, so that both $\boldsymbol{B}$ and $\boldsymbol{C}$ are n-dimensional column vectors. A recirculation term may then be added wherever it is required in $\boldsymbol{F}$ by entering, in the appropriate places, an rQ in $\boldsymbol{B}$, and a 1 in $\boldsymbol{C}$, all other entries being zero. Thus, in Figure 4 above, the addition of rQ to f_{49} is achieved by setting

$$\boldsymbol{B} = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\rQ\\0\\0\\0 \end{bmatrix} \text{ and } \boldsymbol{C} = \begin{bmatrix} 0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}$$

so that

$$\boldsymbol{B}\boldsymbol{C}^{\mathrm{T}} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&rQ&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \tag{19}$$

Clearly, additional recirculation paths can be added by further rQ entries in $\boldsymbol{B}$ and ones in $\boldsymbol{C}$. By substitution in the Sherman–Morrison–Woodbury formula, the inverse of the new matrix $\tilde{\boldsymbol{F}}$ (i.e. with recirculation) becomes, in terms of the original matrix $\boldsymbol{F}$,

$$\tilde{\boldsymbol{F}}^{-1} = \boldsymbol{F}^{-1}\left(\boldsymbol{I} - r\boldsymbol{Q}(1 + rQ\boldsymbol{F}^{-1}_{49})^{-1} \begin{bmatrix} 0\\0\\\vdots\\ \boldsymbol{F}^{-1}_{\text{row4}}\\ \vdots\\0\\0 \end{bmatrix} \right) \tag{20}$$

where the $\boldsymbol{F}^{-1}_{\text{row4}}$ entry is row 4 of the inverse of $\boldsymbol{F}$ and is entered here on row 9. When $\tilde{\boldsymbol{F}}^{-1}$ is in turn substituted in Sandberg's formula to find the local mean age, the $r\bar{\tau}_e/(1 - r)$ add-on term emerges, as in section 2, together with an algebraic expression for the local mean age of each zone in terms of r, Q and v_i. Details of the mathematics are given by Simons[5].

6 Discussion

The most significant feature of this analysis is that recirculation adds the same constant term to all the local mean ages in a space. Thus, in the simple case of section 2 above, taking the local mean age (without recirculation) at the exhaust as $\bar{\tau}_e = 10$ min, and a recirculation of 60%, the add-on is $0.6 \times 10/(1-0.6) = 15$ min, and the mean age throughout the space is raised by this amount. It is perhaps worth noting that, as the recirculation increases, the add-on term increases hyperbolically, rapidly approaching infinity as r approaches 100%. A particularly useful feature of the result is that the pattern of a contour diagram of local mean ages is not affected by recirculation. The shape and position of the contours for all recirculation rates will be the same as the fresh air case; only the values of the contour lines will change. Figure 7 is a typical local mean age contour diagram for a displacement ventilation system, plotted for zero recirculation, and computed as described by Simons and Waters[6]. By reading

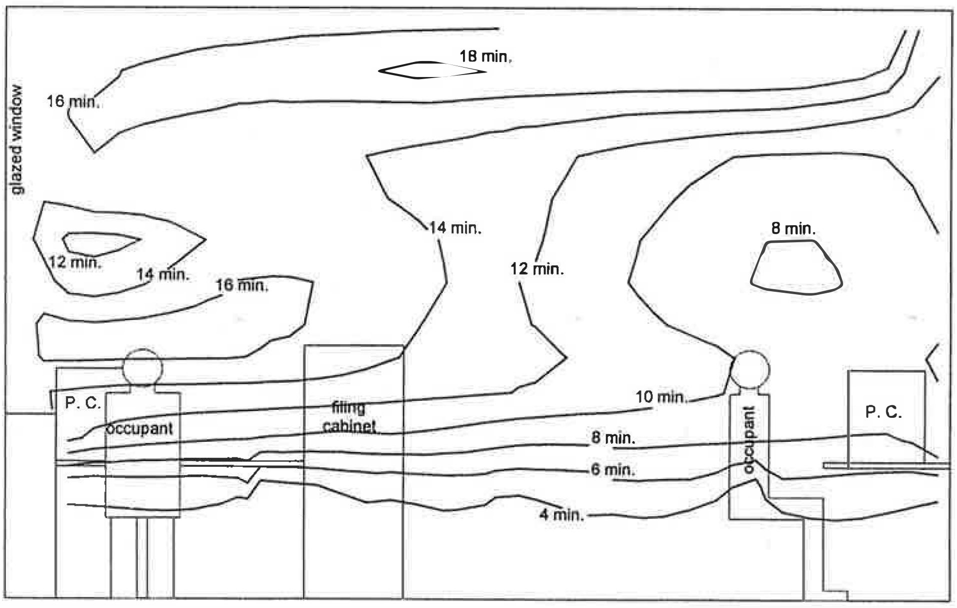

Figure 7 Typical local mean age contour diagram

the value of $\bar{\tau}_e$ from the diagram and applying the add-on formula, the effect of different recirculation rates can immediately be evaluated. Thus, reading from Figure 7, it can be seen that $\bar{\tau}_e \cong 16$ min. In this case, a 70% recirculation rate will raise the local mean age by approximately 37 min.

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