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Natural ventilation induced by combined wind and thermal forces☆

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Abstract

Analytical solutions are derived for calculating natural ventilation flow rates and air temperatures in a single-zone building with two openings when no thermal mass is present. In these solutions, the independent variables are the heat source strength and wind speed, rather than given indoor air temperatures. Three air change rate parameters α, β and γ are introduced to characterise, respectively, the effects of the thermal buoyancy force, the envelope heat loss and the wind force. Non-dimensional graphs are presented for calculating ventilation flow rates and air temperatures, and for sizing ventilation openings. The wind can either assist the buoyancy force or oppose the airflow. For assisting winds, the flow is always upwards and the solutions are straightforward. For opposing winds, the flow can be either upwards or downwards depending on the relative strengths of the two forces. In this case, the solution for the flow rate as a function of the heat source strength presents some complex features. A simple dynamical analysis is carried out to identify the stable solutions.

1. Introduction

Natural forces, i.e. thermal buoyancy and wind, drive natural ventilation of buildings. There are a large number of governing factors in natural ventilation, such as wind speed and direction and its turbulence, the size and position of ventilation openings, the heat sources, the envelope conductance, solar radiation and so on. Accurate prediction of natural ventilation rates at the design stage is often very difficult. The key to accurately predicting natural ventilation rates lies in accurately predicting the two natural forces and their combined effects when necessary.

In most design problems, the thermal buoyancy force and ventilation flow rates are interdependent. Most studies in the past only considered situations where the indoor air temperatures were given (for example, Foster and Down [1]). Etheridge and Sandberg [2] presented some excellent parametric investigations. Their studies used a non-dimensional approach to demonstrate the relative importance of the various parameters. However, they assumed that the indoor air temperatures are known. In reality, the heat sources, wind speed and thermal conductance of the building envelope are known, and the indoor air temperature and ventilation flow rate are derived from these parameters. Simple analytical models were developed for situations when buoyancy acts alone and the air temperature is unknown (see Andersen [3] and Li [4]).

Zhang et al. [5] developed a simple numerical model for combining wind and thermal buoyancy effects in a single-zone building with three openings. The authors also summarised a number of previous studies on analytical solutions of natural ventilation driven by winds alone. It is now possible to simulate multi-zone buildings with multiple openings by combining a thermal

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Nomenclature

A_b area of the bottom opening 'b'
A_j area of the wall j
A_t area of the top opening 't'
A^* effective opening area of a building
B buoyancy flux
C_d discharge coefficient
C_p pressure coefficient
c_p heat capacity of air
c heat capacity of the mass
E effective total heat power
E_i heat from people, equipment and lighting
E_s direct solar heat gain through windows
F intermediate variable
G flow potential
g gravity acceleration
h height between two vertical openings 't' and 'b'
v wind speed

Greek symbols

α thermal air change parameter [Eq. (11)]
β heat-loss air change parameter [Eq. (23)]
γ wind air change parameter [Eq. (23)]
ρ air density

Subscripts

o outside
b bottom opening
t top opening
w wind

analysis program and a multi-zone flow simulation program [6]. The thermal program inputs zonal air temperature into a multi-zone airflow program, and the latter outputs inter-zonal airflow rates back into the thermal program. The calculation iterates until a solution with consistent air temperatures and air flow rates is obtained. These numerical tools can now be used to revisit the work of Etheridge and Sandberg [2] and produce non-dimensional graphs for the simple cases they considered, but with unknown indoor air temperatures. However, an analytical method is preferred before a full numerical study is carried out. It is much easier to carry out a parametric study with analytical solutions than numerical experiments.

Most of today's design codes on natural ventilation still adopt the simple semi-analytical solutions (i.e. the hot air column model and cross-wind ventilation model) as design tools, such as those in CIBSE design guide [7] and BS 5925 [8]. These formulae have been shown to be able to provide reasonable estimates of natural ventilation flow rates in many situations. However, one of the difficulties in using them is that the indoor air temperature must be known prior to the natural ventilation flow rate calculation. One has to adjust the air temperature used after the natural ventilation flow rate is calculated. To obtain a consistent estimate of both flow rate and indoor air temperature is quite often not possible in practice.

This paper derives steady-state analytical solutions for the ventilation rate in a single-zone building with two openings, considering the effect of buoyancy force, wind force, solar radiation and heat conduction loss through the building envelope, and their interactions. The effects of buoyancy force, wind force and heat conduction loss are identified. A dynamic analysis is used to identify the stable solutions when the wind opposes the flow. We assume that there is no thermal mass in the building. If thermal mass is included in the analysis, no analytical solution exists. However, the model can still apply to some practical buildings such as agricultural (e.g. livestock) and industrial buildings with relatively low thermal mass. When ventilation airflow rates are very large, the thermal mass may also be neglected.

2. A simple building model

Consider a simple building with two openings at different vertical levels on opposite walls, as shown in Fig. 1. The heights of the two openings are relatively small. There is an indoor source of heat, E_i, and solar radiation acts on the building via a sol-air temperature for the opaque elements and solar heat gain through windows. The wind force can assist or oppose the thermal buoyancy force. We assume that the indoor air is

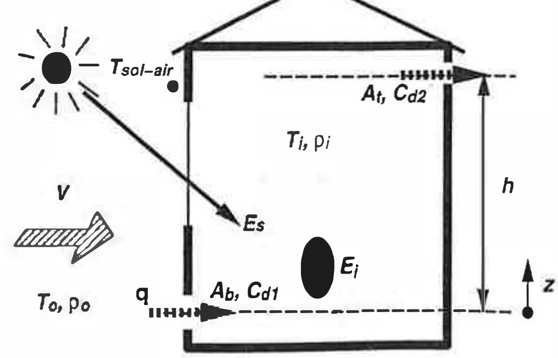

Fig. 1. The general notation for a two-opening building with solar radiation.

fully mixed, i.e. the air temperature is uniform. This assumption is generally not valid for thermal buoyancy force-dominated flow. A non-uniform temperature model will be presented elsewhere. Wind turbulence effects are not included.

A heat balance on the building gives

$$\rho c_p q(T_i - T_o) + \Sigma U_j A_j (T_i - T_{\text{sol-air},\,j}) = E_i + E_s \tag{1}$$

This can be rearranged as

$$\rho c_p q(T_i - T_o) + \Sigma U_j A_j (T_i - T_o) = E \tag{2}$$

where

$$E = E_i + E_s + \Sigma U_j A_j (T_{\text{sol-air},\,j} - T_o) \tag{3}$$

E is referred to as the effective total heat source. It has three parts, the sensible heat source E_i, the direct solar heat gain E_s, and the indirect solar heat gain through walls. Thus, the use of the difference between the sol-air and the outdoor air temperatures allows the heat balance equation to be simply written in terms of the indoor and outdoor air temperature difference as the explicit dependent variable.

As the heat source is assumed to be non-negative, we simply always have

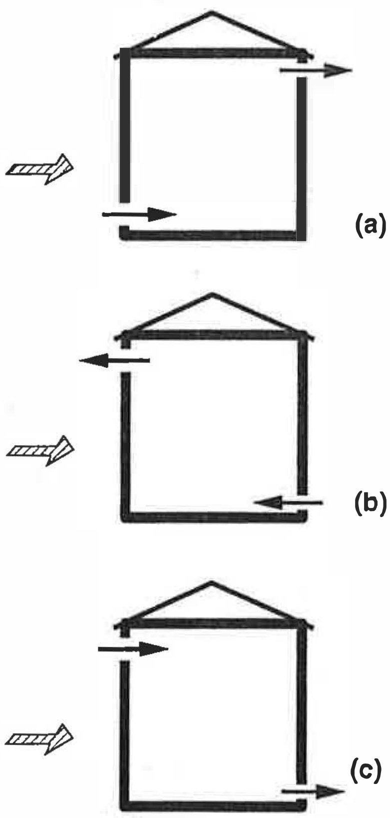

Fig. 2. Three cases considered: (a) assisting wind force; (b) opposing wind force with upward flow; and (c) opposing wind force with downward flow.

$$T_i \geq T_o \tag{4}$$

Three cases are considered, i.e. assisting winds, opposing winds with upward flows and opposing wind with downward flows (see Fig. 2).

3. Natural ventilation driven by thermal force alone

3.1. No heat conduction loss

This is often considered to be the simplest situation in natural ventilation. It is known (see, for example, Li [4] and Bruce [9]) that for a given indoor air temperature, T_i, the ventilation flow rate can be calculated as

$$q = C_d A^* \sqrt{2gh \frac{T_i - T_o}{T_o}} \tag{5}$$

where

$$A^* = \frac{A_t A_b}{\sqrt{A_t^2 + A_b^2}} \tag{6}$$

Both A_b and A_t are free-opening areas. In deriving Eq. (5), the power-law equation is used to describe the relationship between the flow rate and the pressure difference.

In Eq. (5), the C_d values are assumed to be the same for both openings. The effective area can be defined in other ways. With the present definition, the effective area is simply the area of the small opening when the larger opening area approaches infinity. It is fairly easy to write down the formula for the case of non-equal C_d values,

$$A^* = \frac{(C_{dt} A_t)(C_{db} A_b)}{C_d \sqrt{(C_{dt} A_t)^2 + (C_{db} A_b)^2}} \tag{7}$$

where C_d can be taken as equal to either C_{dt} or C_{db}.

It should be mentioned that formula (5) has been presented in many different forms in the literature for the convenience of use by engineers [1]. It is basically a hot air column model. The application of the "hot air column" model requires the knowledge of indoor air temperature, which is again dependent on the ventilation flow rate q. Thus, the airflow rate and the indoor air temperature are coupled.

In order to derive a flow rate solution without the need of air temperature input, we first consider the simple case when a building is perfectly insulated. Assuming the total heat source strength is E, the heat balance equation (2) can be simplified as follows:

$$\rho c_p q(T_i - T_o) = E \tag{8}$$

Substituting Eq. (5) into (8), we derive

$$q = (C_d A^*)^{2/3}(2Bh)^{1/3} \tag{9}$$

where

$$B = \frac{Eg}{\rho c_p T_o} \tag{10}$$

Eq. (9) has been derived by many authors (see, for example, Andersen [3] and Li [4]). Similar results were obtained with other prescribed indoor air temperature profiles, such as the emptying water filling box model of Linden et al. [10] and the emptying air filling box model suggested by Li [4].

For convenience, we define the following α parameter,

$$\alpha = (C_d A^*)^{2/3}(Bh)^{1/3} \tag{11}$$

Eq. (9) becomes

$$q = \sqrt[3]{2}\alpha \tag{12}$$

Thus, the parameter α measures the ventilation flow rate driven by buoyancy force alone. The ventilation flow rate q is simply proportional to α.

Examining the definition of α, it can be seen that the most effective way to increase α and ventilation flow rates is to increase the effective ventilation area, not the heat source strength or the vertical displacement of the two openings. However, this conclusion only applies to buildings with ventilation openings with relatively small heights.

Once the ventilation flow rate is known, the indoor air temperature can be easily calculated by using Eq. (8). We define the following temperature parameter, Q_T, from which the air temperature can be calculated

$$Q_T = 2gh(C_d A^*)^2 \frac{T_i - T_o}{T_o} \tag{13}$$

Thus, for the simplest situation, we have

$$Q_T = q^2 \tag{14}$$

3.2. Heat loss and solar heat gain

When there is heat loss and solar heat gain through walls and windows, we substitute Eq. (5) into (2) and, after some manipulation, we obtain

$$q^3 + 3\beta q^2 - 2\alpha^3 = 0 \tag{15}$$

where α is defined by Eq. (11) and

$$\beta = \frac{\Sigma U_j A_j}{3\rho c_p} \tag{16}$$

β is a very interesting parameter, which measures the effect of conductive heat loss on the ventilation flow rate. It has the same units as α and the ventilation flow rate q. The heat loss air change parameter β may be taken as the equivalent airflow rate resulting from the heat if it is not lost through the building envelope.

Eq. (14) is still valid for calculating the temperature parameter, Q_T.

Three roots for Eq. (15) can be easily obtained. In this paper, the analytical solutions for the cubic equations are obtained using the mathematical software Mathematica,

$$q_1 = -\beta + \frac{\beta^2}{\omega} + \omega$$

$$q_2 = -\beta - \frac{i(-i+\sqrt{3})\beta^2}{2\omega} + \frac{1}{2}i(i+\sqrt{3})\omega$$

$$q_3 = -\beta + \frac{i(i+\sqrt{3})\beta^2}{2\omega} - \frac{1}{2}i(-i+\sqrt{3})\omega \tag{17}$$

where

$$\omega = \left(\alpha^3 - \beta^3 + \sqrt{\alpha^6 - 2\alpha^3\beta^3}\right)^{1/3} \tag{18}$$

The positive real root is the solution for the natural ventilation flow rate q. When $\beta=0$, the root (17) returns to its simplest form of Eq. (12). It is also not difficult to solve Eq. (15) numerically with the knowledge that the solution lies between 0 and $\sqrt[3]{2}\alpha$.

3.3. A natural ventilation graph

Fig. 3 shows the ventilation flow rate, q, and the temperature parameter, Q_T, as a function of the two parameters α and β. The β effect on ventilation flow rate is not linear. As β changes from 0 to 1, there is a big drop in the ventilation flow rate. As β further

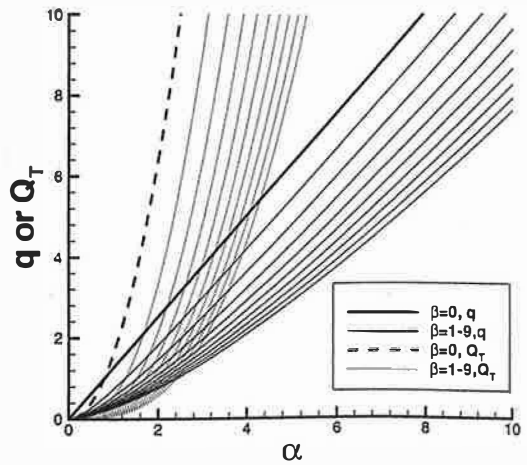

Fig. 3. Ventilation rate and air temperature parameter for thermal buoyancy force alone with heat loss through walls.

increases, the reduction in the ventilation flow rate slows down.

This non-linear effect of β on ventilation flow rate and Q_T/α^3 is explored in Fig. 4, where q/α is plotted against the ratio β/α. As β/α increases from 0, the dimensionless ventilation flow rate q/α first drops sharply from its highest value of $\sqrt[3]{2}$. After β/α reaches about 10, the drop in the relative flow rate is not as fast. Figs. 3 and 4 are effectively identical, except that Fig. 4 shows a wide range of β/α values, but not $\beta/\alpha = 0$.

When $\beta = 0$ (i.e. insulation is perfect), the ventilation flow rate is a linear function of the buoyancy parameter, α, and in contrast, the air temperature parameter, Q_T, increases quadratically as α increases. However, this does not necessarily mean that when the α value in the building increases, the indoor air temperature also increases at a faster rate than the natural airflow rate. In fact, a rise in the temperature parameter corresponds to a drop in indoor air temperature if α increases as a result of increasing the effective ventilation area or the vertical displacement of the two openings. Examining the solutions (11)–(13) when β equals zero, we have the following expression for the indoor air temperature as a function of the independent variables,

$$\frac{T_i - T_o}{T_o} = (C_d A^*)^{-2/3}\left(\frac{E}{\rho c_p T_o}\right)^{2/3}(2gh)^{-1/3} \qquad (19)$$

It can be seen that as the vertical displacement h or the effective ventilation area increases, the indoor air temperature decreases.

For design purposes, Fig. 3 appears to be simple to use. In this paper, we distinguish three different design problems:

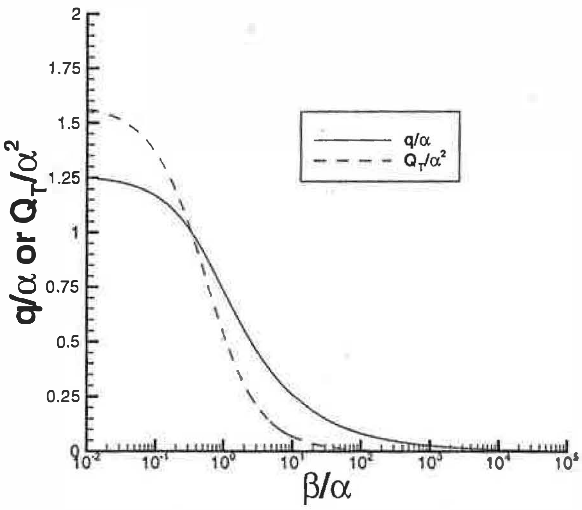

Fig. 4. Non-dimensional plot of the β effect on ventilation flow rate. Both the ventilation flow rate q and the heat loss parameter β are normalized by the buoyancy parameter α.

- Geometry-based design — For a building with known ventilation openings and ventilation flow rates, the air temperature needs to be determined.
- Thermal comfort-based design — For a building with a targeted indoor air temperature, the ventilation openings and the airflow rate need to be determined.
- Indoor air quality-based design — For a building with a targeted airflow rate, the ventilation openings and the indoor air temperature need to be determined.

Fig. 3 is only applicable for buildings with negligible thermal mass and two effective ventilation openings. A procedure for applying Fig. 3 to the above three design problems is suggested as follows:

- Geometry-based design — From the heat source strength and the ventilation opening areas, one determines α. From the U-values and heat transfer areas, β can be calculated. Fig. 3 can then be used to predict the ventilation flow rate and the air temperature. Solar radiation is included as shown in Eq. (2).
- Thermal comfort-based design — From the heat balance equation (2), one can determine the required ventilation flow rate as the indoor air temperature is known. Calculate β, and then determine α in Fig. 3 as both the airflow rate, q, and β are known. The effective ventilation opening area can be determined from α. If one opening area is known, the second can be determined easily.
- Indoor air quality-based design — Calculate β, and then determine α in Fig. 3 as both the airflow rate, q and β are known. The effective ventilation opening area can be determined from α. If one opening area is known, the second can be determined easily. Use Fig. 3 again to determine the temperature parameter, Q_T, which can be used to calculate the indoor air temperature.

The usefulness of Fig. 3 lies in the fact that both airflow rate and air temperature are presented on one graph as a function of both heat source strength and heat loss through the envelope. Fig. 3 applies only when there is no wind.

4. Natural ventilation driven by assisting winds

Depending on the arrangement of ventilation openings, wind can assist or oppose the thermal force in natural ventilation. Two extreme situations are considered here, i.e. fully assisting and fully opposing. These situations occur when there are only two ventilation openings. It appears that no analytical solutions exist for more than two openings.

4.1. Derivation

It is easy to show that the flow rate can be calculated as follows:

$$q = C_d A^* \sqrt{2gh\frac{T_i - T_o}{T_o} + 2\Delta P_w} \tag{20}$$

where

$$\Delta P_w = \tfrac{1}{2} C_{p1} v_1^2 - \tfrac{1}{2} C_{p2} v_2^2 \tag{21}$$

where the subscripts 1 and 2 refer to ventilation openings. ΔP_w is always non-negative. Substituting Eq. (20) into (2), after some manipulation

$$q^3 + 3\beta q^2 - 3\gamma^2 q - 2\alpha^3 - 9\gamma^2\beta = 0 \tag{22}$$

where

$$\gamma = \frac{1}{\sqrt{3}}(C_d A^*)\sqrt{2\Delta P_w} \tag{23}$$

Similarly to α and β, the air change rate parameter γ quantifies the wind effect. It is easy to show that when the wind force acts alone,

$$q = \sqrt{3}\gamma \tag{24}$$

i.e. the ventilation flow rate is proportional to the γ values. The parameter γ is a "scaled" ventilation flow rate induced by wind forces.

If $\beta = 0$, then

$$q = \left(\alpha^3 + \sqrt{\alpha^6 - \gamma^6}\right)^{1/3} + \left(\alpha^3 - \sqrt{\alpha^6 - \gamma^6}\right)^{1/3} \tag{25}$$

If $\alpha = 0$ and $\beta = 0$, we recover Eq. (24).

Three roots for Eq. (22) can be obtained as follows:

$$q_1 = -\beta + \frac{\beta^2 + \gamma^2}{\omega} + \omega$$

$$q_2 = -\beta - \frac{i(-i + \sqrt{3})(\beta^2 + \gamma^2)}{2\omega} + \frac{1}{2} i(i + \sqrt{3})\omega$$

$$q_3 = -\beta + \frac{i(i + \sqrt{3})(\beta^2 + \gamma^2)}{2\omega} - \frac{1}{2} i(-i + \sqrt{3})\omega \tag{26}$$

where

$$\omega = \left[\alpha^3 - \beta^3 + 3\beta\gamma^2 + \sqrt{\alpha^6 - 2\alpha^3(\beta^3 - 3\beta\gamma^2) - (-3\beta^2\gamma + \gamma^3)^2}\right]^{1/3} \tag{27}$$

Eq. (20) can be used to calculate the temperature parameter, Q_T

$$q^2 = Q_T + 3\gamma^2 \tag{28}$$

There are many graphical ways to present the ventilation flow rate as a function of the three parameters α, β and γ. Examination of Eq. (22) shows that the solution can be easily presented in a non-dimensional graph.

The solutions are presented in the following forms

$$\frac{q}{\alpha} = f\left(\frac{\gamma}{\alpha}, \frac{\beta}{\alpha}\right) \tag{29}$$

and

$$\frac{q}{\gamma} = f\left(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}\right) \tag{30}$$

Equally, the air temperature parameter, Q_T, can be presented in the following forms

$$\frac{Q_T}{\alpha^2} = f\left(\frac{\gamma}{\alpha}, \frac{\beta}{\alpha}\right) \tag{31}$$

and

$$\frac{Q_T}{\gamma^2} = f\left(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}\right) \tag{32}$$

The important non-dimensional ratios α/γ or γ/α are relative measures of the two driving forces.

4.2. Two ventilation graphs

The ventilation flow rate and temperature parameter are presented as a function of the buoyancy parameter and the heat loss parameter in Fig. 5. Fig. 5 integrates the effects of wind force, thermal buoyancy and heat loss through the building envelope on flow rate and air temperature into one graph.

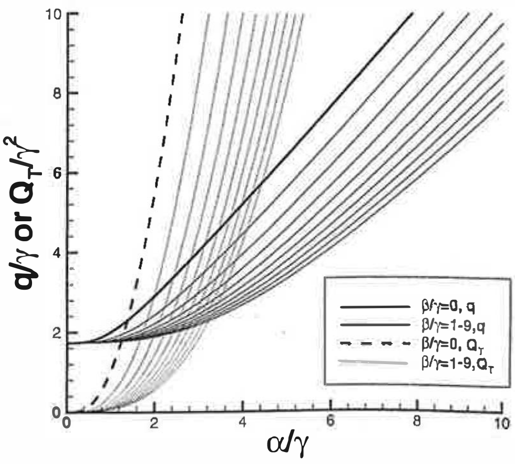

Fig. 5. Ventilation rate and air temperature parameter as a function of the thermal buoyancy parameter for combined forces with assisting winds and heat loss through walls.

There are two main differences between Fig. 5 (with wind) and Fig. 3 (without wind).

The first difference is that assisting winds always increase the ventilation flow rate. The effect of assisting wind forces is to "lift up" the total ventilation flow rate in Fig. 5, compared to Fig. 3. At $\alpha=0$, the ratio q_v/γ is $\sqrt{3}$. The reduction effect of the heat loss parameter β also depends on γ, which means that Fig. 5 is not geometrically similar to Fig. 3. When $\beta=0$, the ventilation flow rate with wind is no longer a linear function of the buoyancy parameter.

Secondly, the effect of β on ventilation reduction is also a function of α and γ. When α/γ is less than one, the flow rates for different β values are all close to $\sqrt{3}\gamma$. This is obvious, as the flow is wind-dominated when α/γ is close to zero.

In Fig. 6, the ventilation flow rate and the air temperature parameter are plotted against the wind force parameter, γ/α.

When the wind force is much stronger than the thermal force, i.e. $\gamma/\alpha \gg 1$, the ventilation flow rate is independent of the heat loss parameter. The system converges to the straight line $q=\sqrt{3}\gamma$. When $\gamma/\alpha < 2$, the system converges much more quickly to the straight line $q=\sqrt{3}\gamma$ as β increases. This is because the larger the heat loss, the less the buoyancy force, and the closer the combined driving force is to the wind force.

The air temperature parameter, Q_T, decreases as γ/α increases. It is interesting that the ratio Q_T/α^2 is almost independent of γ/α, when $\beta/\alpha \gg 1$. This can be explained by further examining Eqs. (22) and (28). From these two equations, we obtain

$$\frac{Q_T}{\alpha^2}=\frac{2}{3\frac{\beta}{\alpha}+\frac{q}{\alpha}} \tag{33}$$

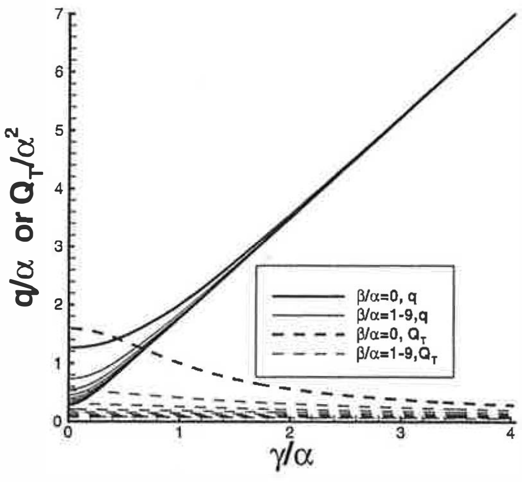

Fig. 6. Ventilation rate and air temperature parameter as a function of the wind parameter for combined forces with assisting winds and heat loss through walls.

Considering $(\beta/\alpha) \gg 1$, the ventilation flow is wind-dominated. The above equation can be approximated as follows:

$$\frac{Q_T}{\alpha^2}=\frac{2}{3\frac{\beta}{\alpha}+\sqrt{3}\frac{\gamma}{\alpha}} \tag{34}$$

It can be seen that when $(\beta/\alpha) \gg (\gamma/\alpha)$, the solution for Q_T/α^2 is only a function of β/α. However, when γ/α approaches infinity, Q_T/α^2 approaches zero, which is not shown in Fig. 6.

4.3. Use for design purposes

Fig. 5 is very useful when both wind and buoyancy forces coexist and are comparable in a building. A procedure for applying Fig. 5 to the three design problems is as follows:

- Geometry-based design — From the heat source strength and the ventilation opening areas, one determines α. From the U-values and heat transfer areas, β can be calculated. The wind parameter γ can be calculated from the ventilation openings and the wind pressure. Fig. 5 can then be used to find the ventilation flow rate and the air temperature.
- Thermal comfort-based design — From the heat balance equation (2), one can determine the required ventilation flow rate as the indoor air temperature is known. Calculate the wind pressure, ΔP_w, and use the flow rate equation (20) to determine the effective ventilation area. If one opening area is known, and the second one can be determined easily. Calculate α, β and γ to check the system performance in Fig. 5 to see the balance of the natural ventilation forces.
- Indoor air quality-based design — From heat balance equation (2), one can determine the indoor air temperature as the required ventilation flow rate is known. Calculate the wind pressure, ΔP_w, and use the flow rate equation (20) to determine the effective ventilation area. If one opening area is known, and the second one can be determined easily. Calculate α, β and γ to check the system performance in Fig. 5 to see the balance of the natural ventilation forces.

5. Natural ventilation driven by opposing winds

5.1. Derivation

The flow direction considered so far is always upward. When the wind force opposes the thermal buoyancy force, the flow can be either upward or downward, depending on the relative strengths of the forces. Again, it can be easily derived that

$$q = C_d A^* \sqrt{\left| 2gh \frac{T_i - T_o}{T_o} - 2\Delta P_w \right|} \tag{35}$$

The heat balance equation is the same as Eq. (2). Applying the heat balance equation, we obtain

$$q^3 + 3\beta q^2 = | 2\alpha^3 - 3\gamma^2 q - 9\gamma^2 \beta | \tag{36}$$

For upward flows, the buoyancy force is stronger and $2\alpha^3 > 3\gamma^2 q + 9\gamma^2\beta$. We have

$$q^3 + 3\beta q^2 + 3\gamma^2 q - 2\alpha^3 + 9\gamma^2\beta = 0 \tag{37}$$

Three roots for Eq. (37) are

$$q_1 = -\beta + \frac{\beta^2 - \gamma^2}{\omega} + \omega$$

$$q_2 = -\beta - \frac{i(-i + \sqrt{3})(\beta^2 - \gamma^2)}{2\omega} + \frac{1}{2} i(i + \sqrt{3})\omega$$

$$q_3 = -\beta + \frac{i(i + \sqrt{3})(\beta^2 - \gamma^2)}{2\omega} - \frac{1}{2} i(-i + \sqrt{3})\omega \tag{38}$$

where

$$\omega = \left[\alpha^3 - \beta^3 - 3\beta\gamma^2 + \sqrt{\alpha^6 - 2\alpha^3(\beta^3 + 3\beta\gamma^2) + (3\beta^2\gamma + \gamma^3)^2} \right]^{1/3} \tag{39}$$

For downward flows, the wind force is stronger and $2\alpha^3 < 3\gamma^2 q + 9\gamma^2\beta$. We have

$$q^3 + 3\beta q^2 - 3\gamma^2 q + 2\alpha^3 - 9\gamma^2\beta = 0 \tag{40}$$

Three roots for Eq. (40) are

$$q_1 = -\beta + \frac{\beta^2 - \gamma^2}{\omega} + \omega$$

$$q_2 = -\beta - \frac{i(-i + \sqrt{3})(\beta^2 + \gamma^2)}{2\omega} + \frac{1}{2} i(i + \sqrt{3})\omega$$

$$q_3 = -\beta + \frac{i(i + \sqrt{3})(\beta^2 + \gamma^2)}{2\omega} - \frac{1}{2} i(-i + \sqrt{3})\omega \tag{41}$$

where

$$\omega = \left[-\alpha^3 - \beta^3 + 3\beta\gamma^2 + \sqrt{\alpha^6 + 2\alpha^3(\beta^3 - 3\beta\gamma^2) - (-3\beta^2\gamma + \gamma^3)^2} \right]^{1/3} \tag{42}$$

Eq. (35) is used to derive the formula for calculating the air temperature parameter, Q_T

$$q^2 = | Q_T - 3\gamma^2 | \tag{43}$$

5.2. Solution behaviour

Although the analytical solutions of the governing equations are given above, they do not easily reveal the behaviour of the flow rate and air temperature. Simple analyses of the solution behaviours can be done as is shown in the Appendix. An analysis of the behaviour of the flow rate as a function of α and γ reveals that the general form of the solutions must be as shown in Fig. 7 for $\beta = 0$ and Fig. 8 for $\beta \neq 0$.

This shows some interesting features. For example, consider a very low value of α, so that the flow is definitely downward (see Figs. 7a and 8a). As α increases, we move to the right along the downward flow curve, and the flow rate decreases. However, when α reaches a critical value, α_B, denoted by point B in Fig. 3, an interesting phenomenon occurs. If α increases slightly, the flow direction reverses to upward flow and the flow rate drops to a lower value, denoted by point E. If α then increases further, the flow rate increases, as would be expected. However, if at *E*, α now decreases, the upward flow rate can decrease to zero at $\alpha = \alpha_A$, at point A. If α further decreases, then the flow reverses to downward, and the ventilation flow rate jumps to point C. Furthermore, for $\alpha_A < \alpha < \alpha_B$, there appears to be three possible flow rates for a given value of α: two downward flows and one upward flow.

Similar features exist for Figs. 7b–d and 8b–d.

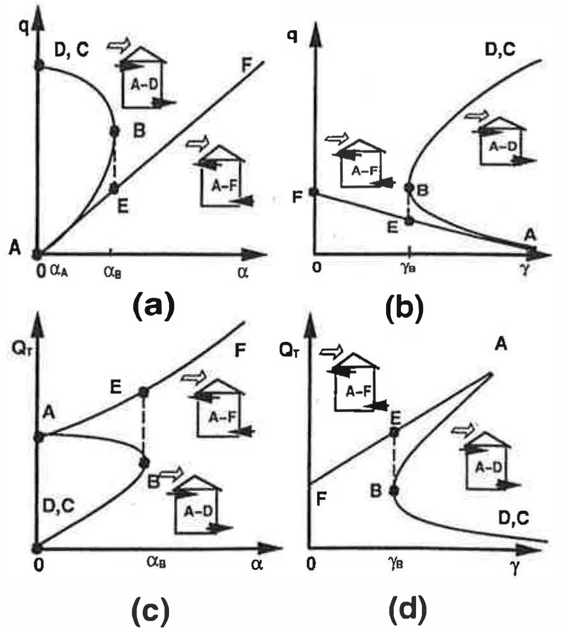

Fig. 7. Analytical sketches of the solution behaviour when $\beta = 0$.

Finally, the state of the system represented by the curve A–B is unusual. Here the flow direction is downward, but an increase in α results in an increase in the flow rate. This is counter-intuitive: since the wind is opposing the buoyancy force and is stronger (i.e. downward flow), one would expect that an increase in the buoyancy force would result in a decrease in the flow rate, not an increase. Thus, the curve A–B may be a non-physical or unstable region, even though points on this curve do satisfy the original equations. It is not clear what other constraints can be applied to show that A–B is non-physical. An attempt will be made in the next section to apply a dynamic analysis to show that the solutions of A–B are, in fact, not stable.

5.3. Four ventilation graphs

Two non-dimensional graphs are presented in Figs. 9 and 10 with the ventilation flow rate as a function of the buoyancy parameter and the wind parameter, respectively. For comparison purposes, the assisting wind solutions are also presented in the two graphs with dotted or dashed lines.

Fig. 9 shows that the solutions for the opposing flow are very complex. Let us exclude the A–B curves for all β values. If we look at the two extreme situations, i.e. buoyancy-dominated flows and wind-dominated flows, the picture becomes a bit clearer. When α/γ is less than say $0.5\alpha_B$, the non-dimensional ventilation flow rate q/γ converges to more or less a constant ($\sqrt{3}$). When α/γ is much greater than α_B, the non-dimensional ventilation flow rate q/γ converges to the solutions for thermal buoyancy force alone (refer to the solutions in Fig. 3).

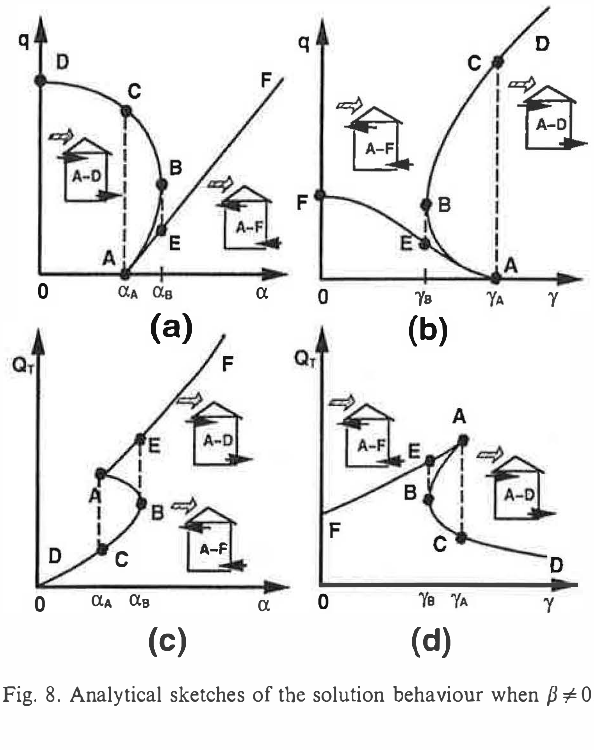

Fig. 8. Analytical sketches of the solution behaviour when $\beta \neq 0$.

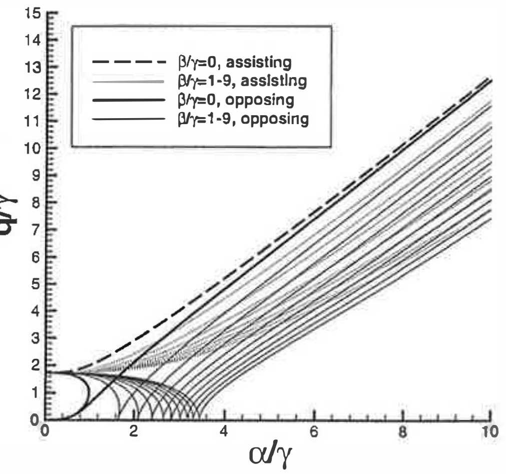

Fig. 9. Ventilation flow rate as a function of the thermal buoyancy parameter for combined forces with heat loss through walls — both opposing and assisting winds.

A similar analysis can also be carried out for Fig. 10. But let us find other interesting but obvious features in Fig. 10. As β increases, the system converges much more quickly to the straight line $q = \sqrt{3}\gamma$. This is because of the fact that the larger the heat loss, the less the buoyancy force. One can also see that there is a certain symmetry along the $q = \sqrt{3}\gamma$ line when comparing the solutions for assisting wind and for opposing wind. For assisting winds, as the heat loss parameter increases, the buoyancy force decreases and the combined driving force also decreases. Thus, the resulting flow rate decreases. For opposing winds, the buoyancy force also decreases when the heat loss parameter increases. However, in contrast, the combined force decreases as the buoyancy force opposes the

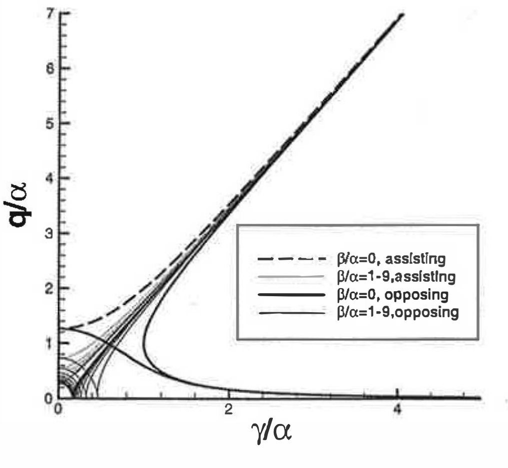

Fig. 10. Ventilation flow rate as a function of the wind parameter for combined forces with heat loss through walls — both opposing and assisting winds.

flow. As a result, the ventilation flow rate increases. Obviously, when $\gamma=0$, the opposing and assisting solutions converge to the same points.

According to the above discussion, there are two turning points in the flow system, α_A and α_B, which will be referred to as the "A turning point" and the "B turning point". At a turning point, the flow direction reverses.

The A turning point is in fact a downward flow turning point. Starting at point F, the flow is upwards. If either the buoyancy force decreases or the wind force increases, the flow rate decreases following the F–A curve. When the flow reaches the A turning point, the flow direction reverses and a downward flow is achieved at point C.

The B turning point is in fact an upward flow turning point. Starting from point D, the flow is downward. As either the buoyancy force increases or the wind force decreases, the flow rate decreases until it reaches point B. When the driving forces continue to change in the same direction, the flow direction reverses to upward flow at Point E.

The situation when the flow system is perfectly insulated is very interesting and worthy of more discussion. Here $\beta=0$. The B turning point always occurs when $\alpha=\gamma$, i.e. when the two driving forces are balanced. The A turning point always occurs when the buoyancy force does not exist or the wind force is infinite.

It is also interesting to note that the two turning points get much closer when the heat loss parameters are very large relative to either α or β.

The turning points also represent the minimum upward ventilation rates that the system can achieve for a given β. In theory, the upward ventilation flow rate at the A turning point, q_A, is always zero. However, the upward ventilation flow rate at the B turning point, q_B, approaches zero as β increases.

In Fig. 9, any two iso-β/γ lines can intersect. The intersections indicate that with the same α/γ value, two different β/γ values can result in the same ventilation flow rate. At any intersection point, the smaller β/γ corresponds to a smaller heat loss through the building envelope. The thermal buoyancy force dominates the flow, and the ventilation flow is upwards. For the larger β/γ, there is more heat loss through the envelope. The thermal force is weaker and the wind force dominates the flow. The flow is downwards.

Similarly, two non-dimensional graphs are presented in Figs. 11 and 12 with the temperature parameter as a function of the buoyancy parameter and the wind parameter, respectively. Again, for comparison purposes, the assisting wind solutions are also presented in the two graphs with dotted or dashed lines. The following conclusions can be drawn:

- The temperature parameter for the opposing winds is always larger than that for the assisting winds for the same set of parameters. This is because the ventilation flow rate for opposing winds is always less than that for assisting winds.
- For both buoyancy-dominated and wind-dominated flows, the solutions for both the opposing and assisting winds are close to identical.

6. Dynamics of natural ventilation

The opposing wind case presents an interesting question. For a certain range of α values, there appear to be three possible flow rates for a given value of α: two downward flows and one upward flow, depending on whether α has been increasing or decreasing, i.e. the system exhibits hysteresis. The question is, which of the three solutions are physical? Discussion in the pre-

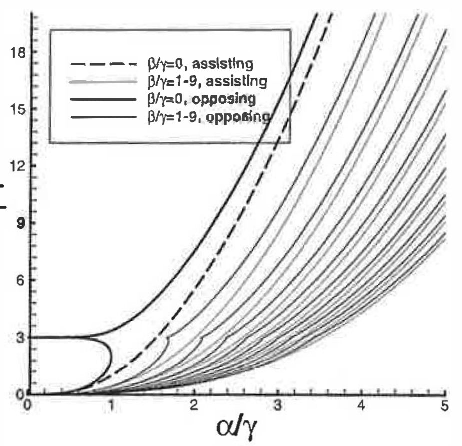

Fig. 11. Temperature parameter as a function of α for natural ventilation with combined forces and heat loss through walls.

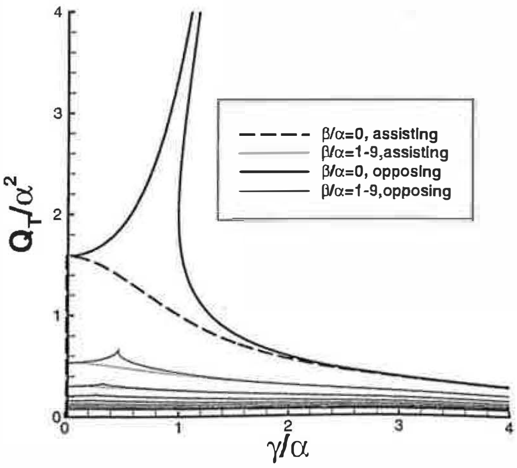

Fig. 12. Temperature parameter as a function of γ for natural ventilation with combined forces and heat loss through walls.

vious section indicated that the curve A–B may not be physical or stable. A dynamic analysis is used here to prove that the curve A–B is indeed unstable.

An example of the three solutions is given in Fig. 13. The solutions are for a building with perfect thermal insulation, i.e. $\beta=0$, and with $\alpha=0.9$, $\gamma=1$. All three solutions satisfy both the heat balance equation and the simplified momentum equation.

Consider an unsteady heat transfer situation in the building. We assume there is an idealised mass M in the building. The temperature of the mass is the same as the air temperature.

The heat balance equation becomes

$$cM\frac{\partial(T_i - T_o)}{\partial t} + \rho c_p q(T_i - T_o) + \Sigma U_j A_j (T_i - T_o) = E \quad (44)$$

where the outdoor air temperature is assumed to be constant. Substituting the flow equation (35) into the new heat balance equation, we have, for upward flows

$$2\omega q\frac{\partial q}{\partial t} = -(q^3 + 3\beta q^2 + 3\gamma^2 q - 2\alpha^3 + 9\gamma^2\beta) \quad (45)$$

and for downward flows

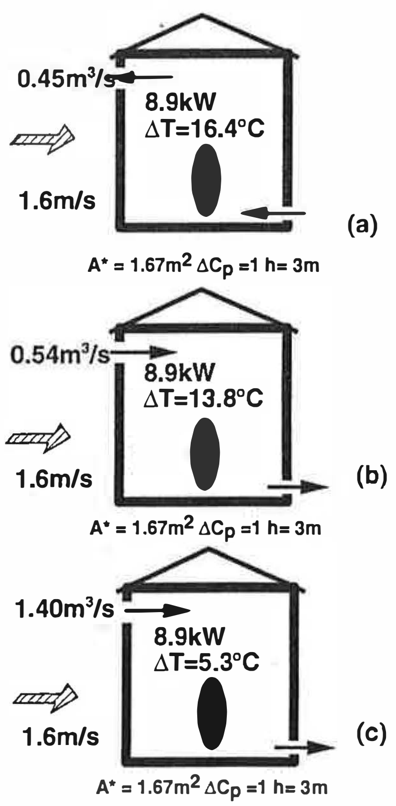

Fig. 13. Three possible flow states in a building with the same ventilation parameters.

$$2\omega q\frac{\partial q}{\partial t} = -(q^3 + 3\beta q^2 - 3\gamma^2 q + 2\alpha^3 - 9\gamma^2\beta) \quad (46)$$

ω is a thermal mass parameter, defined as

$$\omega = \frac{Mc}{\rho c_p} \quad (47)$$

We can write the above equations as

$$2\omega\frac{\partial q}{\partial t} = F(q, \alpha, \beta, \gamma) \quad (48)$$

where

$$F = -\left[q^2 + 3\beta q - 3\gamma^2 + \frac{1}{q}(2\alpha^3 - 9\gamma^2\beta)\right] \quad (49)$$

for upward flows, and

$$F = -\left[q^2 + 3\beta q - 3\gamma^2 - \frac{1}{q}(2\alpha^3 - 9\gamma^2\beta)\right] \quad (50)$$

for downward flows.

This is a one-dimensional dynamic system. Theoretically, the dynamics are entirely determined by the nature and the position of the so-called fixed point of F, which are the solutions of the ventilation flow rates [11].

The system can also be written as

$$2\omega\frac{dq}{dt} = -\frac{\partial G}{\partial q} \quad (51)$$

where $G(q, \alpha, \beta, \gamma) = -\int F\, dq$.

The fixed points of F (ventilation rate solutions) become the stationary points of G. A local minimum of G is a stable fixed point (an attractor), a local maximum is unstable (a repellor), and an inflection point with horizontal tangent is semi-stable (a vague attractor).

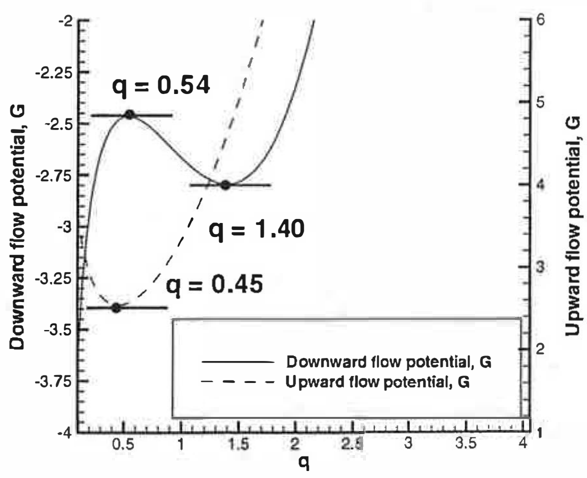

Fig. 14. Flow potential for the opposing flows.

For upward flows

$$G = \tfrac{1}{3}q^3 + \tfrac{3}{2}\beta q^2 + 3\gamma^2 q - (2\alpha^3 - 9\gamma^2\beta)\log q \quad (52)$$

For downward flows

$$G = \tfrac{1}{3}q^3 + \tfrac{3}{2}\beta q^2 - 3\gamma^2 q + (2\alpha^3 - 9\gamma^2\beta)\log q \quad (53)$$

The flow potentials are presented in Fig. 14. While the downward flow of 1.40 m^3/s and the upward flow of 0.45 m^3/s are stable, the downward flow of 0.54 m^3/s is not stable, and this flow is indeed located on the A–B curve. Similarly, the same analysis can be done for other points on the A–B curve to prove that the A–B curve is not stable.

7. Conclusions

Analytical solutions are derived for natural ventilation in a single-zone building with two openings. Three air change parameters are introduced: the thermal air change parameter α, the wind air change parameter γ and the heat loss air change parameter β. It is believed that the simple ventilation graphs presented can be used for design purposes. The following conclusions can be drawn:

- The ventilation flow rate is simply proportional to α or γ when each driving force acts alone.
- The effect of heat loss through the building envelope is very significant. The ventilation flow rate sharply drops when β increases from 0. The change in ventilation flow rates slows down when β further increases. When the wind force is present, the heat loss effect also interacts with wind-induced flows. This is due to the fact that heat loss depends on the indoor air temperature, which is in turn controlled by ventilation.
- When the wind force opposes the thermal buoyancy, for a certain range of α values there appears to be three possible flow rates for a given value of α: two downward flows and one upward flow, depending on whether α has been increasing or decreasing, i.e. the system exhibits hysteresis.
- A dynamic analysis shows that the A–B curve in the non-dimensional graphs of the natural ventilation solutions is not stable.

It should be mentioned that the analytically derived hysteresis behaviour for opposing wind has not been experimentally verified. An experimental program is currently being developed and the results will be published elsewhere.

Acknowledgements

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Appendix. Analysis of natural ventilation

The behaviour of the analytical solutions for the flow equation in the case of opposing winds is not obvious. Here we carry out a simple analysis of the behaviour of the flow rate as a function of the buoyancy parameter, α.

For upward flows, the buoyancy force is stronger than the wind force, $2\alpha^3 > 3\gamma^2 q + 9\gamma^2\beta$. We have the following equation

$$q^3 + 3\beta q^2 + 3\gamma^2 q - 2\alpha^3 + 9\gamma^2\beta = 0 \quad (54)$$

It is convenient to solve for the buoyancy parameter, α

$$\alpha = \left(\frac{q^3 + 3\beta q^2 + 3\gamma^2 q + 9\gamma^2\beta}{2}\right)^{1/3} \quad (55)$$

Thus, at $q = 0$ (point A in Fig. 8a), we obtain

$$\alpha = \left(\frac{9\gamma^2\beta}{2}\right)^{1/3} \quad (56)$$

We also have

$$\frac{d\alpha}{dq} = \frac{q^2 + 2\beta q + \gamma^2}{2\alpha^2} \quad (57)$$

Again, at $q = 0$, we have

$$\left.\frac{d\alpha}{dq}\right|_{q=0} = \frac{\gamma^2}{2\left(\frac{9\gamma^2\beta}{2}\right)^{2/3}} > 0 \quad (58)$$

The above analysis shows that the flow rate behaviour for upward flow must be like A–F in Fig. 8a.

For downward flows, the buoyancy force is weaker than the wind force, $2\alpha^3 < 3\gamma^2 q + 9\gamma^2\beta$. We have

$$q^3 + 3\beta q^2 - 3\gamma^2 q + 2\alpha^3 - 9\gamma^2\beta = 0 \quad (59)$$

Again, one can solve for α,

$$\alpha = \left(\frac{-q^3 - 3\beta q^2 + 3\gamma^2 q + 9\gamma^2\beta}{2}\right)^{1/3} \quad (60)$$

and

$$\frac{d\alpha}{dq} = \frac{-q^2 - 2\beta q + \gamma^2}{2\alpha^2} \quad (61)$$

At $q = 0$, we have

$$\left.\frac{d\alpha}{dq}\right|_{q=0} = \frac{\gamma^2}{2\left(\frac{9\gamma^2\beta}{2}\right)^{2/3}} > 0 \quad (62)$$

We can also identify a point B, where $(d\alpha/dq) = 0$, at $q^2 + 2\beta q - \gamma^2 = 0$. That is

$$q_B = \sqrt{\beta^2 + \gamma^2} - \beta \quad (63)$$

Substituting q_B into Eq. (60), it can be shown that $\alpha_B > 0$ for the downward flow. When $\alpha = (9\gamma^2\beta/2)^{1/3}$, the flow rate equation has two solutions

$$q = 0$$

$$q^2 + 3\beta q - 3\gamma^2 = 0 \quad (64)$$

As the ventilation flow rate is defined to be always positive, at point C we have

$$q_C = \frac{-3\beta + \sqrt{9\beta^2 + 12\gamma^2}}{2} \quad (65)$$

It can also be shown that $q_C > q_B$ and $\alpha_B > \alpha_A$.

At point D, from Eq. (61) we obtain $(d\alpha/dq) = \infty$ if $-q_D^2 - 2\beta q_D + \gamma^2 \neq 0$.

The above analysis shows that the ventilation flow rate behaviour as a function of the buoyancy parameter α must be as shown in Fig. 8a.

It appears that the system exhibits hysteresis. Starting from point D (downward flow and no buoyancy force), as α increases, we reach point B. As α continues to increase, the solution implies that the flow direction reverses suddenly to the upward flow and the flow rate drops to point E. As α further increases, the upward flow rate also increases.

If we start at a point F where the buoyancy force is dominant (i.e. very large α), and the flow is upwards, thus as α decreases, we move down the upward flow curve, pass point E and reach A, where $q = 0$. As α further decreases, no solution exists for upward flow and the system jumps to point C, and the flow direction reverses. The flow rate also jumps from zero to q_C. Further, reducing the buoyancy force to zero leads to point D where the flow will be completely dominated by the wind force at point D.

In fact, we never reach the curve A–B by simply increasing and decreasing α. However, if we reach point B by increasing α along the downward flow curve, what happens if we then reduce α? Will the system enter the B–A curve?

A similar analysis can be carried out to analyse the ventilation flow rate behaviour as a function of the wind parameter γ and the behaviour of the air temperature parameter, as shown in Figs. 7 and 8.

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