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Conformal mapping of a solution to a mixed boundary problem for soil gas flow

Andrew Cripps*,[1]

Building Research Establishment, Watford, WD2 7JR, U.K.

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Abstract

The principal soil gases of current concern to building are radon and landfill gas. The flow of these is generally considered to be dominated by viscous flow driven by pressure differences. This paper presents results for the pressure-driven flow of gas for problems relating to a building with a bare soil floor, for example below a suspended timber floor. This paper builds on a previous paper by mapping the solution to a mixed boundary problem onto another geometry. In a third paper these results will be compared with an analytical result from elsewhere and an experiment. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

This paper develops the content of another paper [1], and it is expected that they will be read together. Therefore the introduction to the driving forces, the background theory and the justification for the work is not repeated here.

Conformal Mapping is a process whereby a solution to a problem found in one co-ordinate set is transferred to another co-ordinate set. This gives a solution to a problem which we may not have been able to solve in another way, or may be simpler than a direct method. It is discussed in most standard textbooks of mathematics for graduate scientists and engineers, for example [2] or [3].

Therefore, because we have solved a mixed boundary-value problem in the previous paper, we can use that solution to find the solution to another mixed boundary-value problem by mapping the first onto a different co-ordinate set. The first step is to tackle a problem with thin foundation walls extending into the ground. The walls are assumed to allow no flow through them, which is a reasonable approximation in permeable soils. Then the method is extended to problems where the walls have thickness as well as depth. This problem is more general than the first and only slightly more difficult to evaluate. However, it is easier to make approximations to the result when the walls are thin.

The methods used predict both the flow rate caused by the defined pressure distribution and the pressure field in the soil. The result for the flow rate is the simpler, and will be the most useful. The description of the pressure field involves numerical integrations, and is therefore harder to use. However it will be useful for model inter-comparisons and understanding experimental data.

Because the equation solved, Laplace's Equation, is applicable to many problems, these results may be of interest to those working in other scientific areas. Flow of heat is the most obvious example, because heat loss through ground floors is closely analogous the flow of gas through the soil and into buildings.

2. Pressure field for house with bare soil floor and thin footings

2.1. Problem definition

The mixed boundary configuration of the previous paper [1] gives the solution to a problem in which the boundary condition was defined along a horizontal surface. The majority of buildings have walls which extend into the ground, so it would be useful to be able to solve this type of problem as well. As a first step we consider a thin wall extending into the ground as shown in Fig. 1: Note that d and c are positive, real numbers.

The problem then is to find the transformation that maps points from fig. 2 of [1] (shown as Fig. 2 here), onto Fig. 1, so that we can find the pressure field for Fig. 1 without further solving the basic equations. The points

* E-mail: andrew.cripps@burohappold.com

[1] Present address: Buro Happold, 17 Newman Street, London W1P 3HD, U.K.

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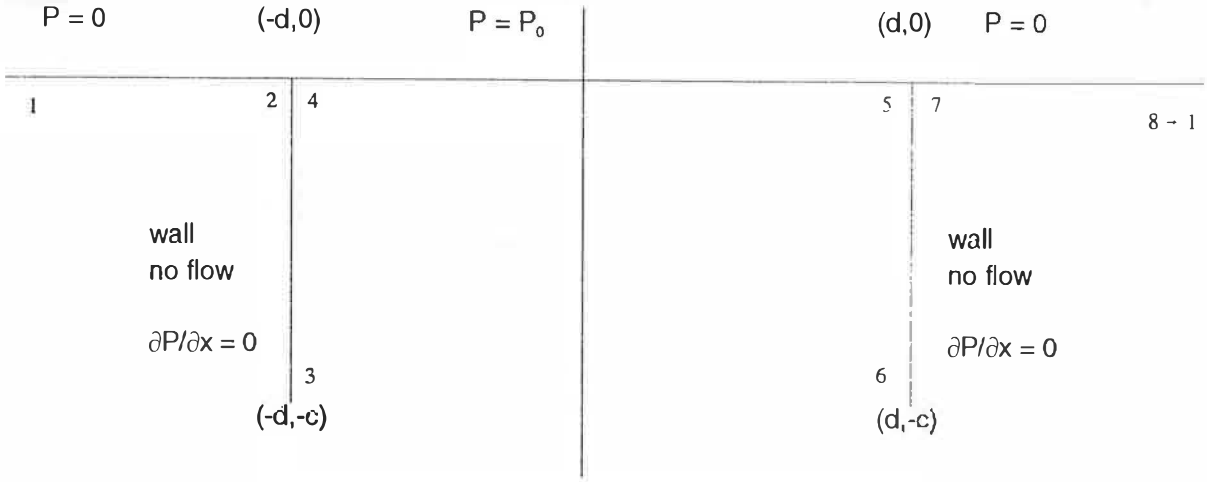

Fig. 1. Diagram for the mapped pressure field problem, the $Z(X, Y)$ plane.

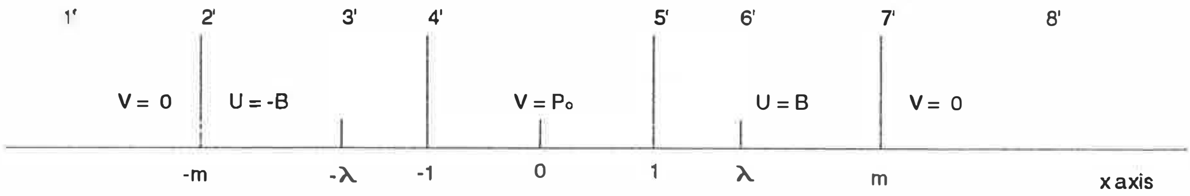

Fig. 2. Diagram of the plane mapped from, the $z(x, y)$ plane.

1′ to 7′ on Fig. 2 are the points in the $z(x, y)$ plane which will transform to the $Z(X, Y)$ plane as points 1 to 7.

2.2. Definition of the transformation

The mapping used is called the Schwarz–Christoffel transformation [4]. It gives the transformation which maps a plane onto a closed polygon, if the positions of the corners and angles at each corner are known. It is

$$Z(X, Y) = f(z) = a\int_{z_0}^{z} \sum_{j=1}^{n} (z' - e_j)^{(\alpha_j/\pi)-1}\, dz' + b \tag{1}$$

where a, b and z_0 are constants to be determined, but we can choose $b = z_0 = 0$; α_j are the angles at the points in the Z-plane; e_j are the positions of the corresponding points in the z-plane; n is the number of points on the x-axis, here 7 since points 1 and 8 are the same.

In this case the angles α_j are $\pi/2$, 2π, $\pi/2$, $\pi/2$, 2π, $\pi/2$ respectively for the points 2 to 7. The e_j are the points on the Y-axis $-m$, $-\lambda$, -1, 1, λ, m.

Hence the transformation needed is

$$Z = f(z) = am\cdot\left(\int_0^z \frac{(z'^2-\lambda^2)\,dz'}{\sqrt{[(z'^2-1)\cdot(m^2z'^2-1)]}}\right). \tag{2}$$

This can usefully be rearranged to

$$Z = f(z) = am^2\cdot\left(\int_0^z \frac{\sqrt{\left(\frac{z'^2}{m^2}-1\right)}}{\sqrt{(z'^2-1)}}\cdot dz' + \left(1-\frac{\lambda^2}{m^2}\right)\cdot\int_0^z \frac{dz'}{\sqrt{\left[\left(\frac{z'^2}{m^2}-1\right)\cdot(z'^2-1)\right]}}\right). \tag{3}$$

This expression contains commonly occurring elliptic integrals which have to be calculated numerically for nearly all values of m and z. The values are found in tables, for example [5], or by numerical integration. Hence Eqn (3) can be written in a shorter form using the notation for elliptic integrals,

$$Z = am^2\cdot\left[E\left(\frac{1}{m}, z\right) + \left(1-\frac{\lambda^2}{m^2}\right)\cdot F\left(\frac{1}{m}, z\right)\right] \tag{4}$$

where $F(1/m, z)$ is the elliptic integral of the first kind, and $E(1/m, z)$ is the elliptic integral of the second kind.

2.3. Finding the parameters in the transformation

There are then three unknowns remaining, a, λ and m. However, there are three points where the result of the transformation is known:

$$f(1) = d \tag{5}$$

corresponds to points 5 and 5′,

$$f(\lambda) = d - ic \tag{6}$$

corresponds to points 6 and 6′, and

$$f(m) = d \tag{7}$$

corresponds to points 7 and 7′.

We could equally well have used points 2, 3 and 4, but the symmetry used earlier means that no further information can be gained from that. Using Eqs. (5)–(7) with the expression for the transformation (3) or (4) defines the relationships between the parameters, m, λ, a, c and d.

Using Eqn (5) and (4) we obtain a relationship between d and a:

$$Z = f(1) = am^2\left[E\left(\frac{1}{m}, 0 \to 1\right) + \left(1 - \frac{\lambda^2}{m^2}\right)\cdot F\left(\frac{1}{m}, 0 \to 1\right)\right] = d \tag{8}$$

where the E and F terms are integrated from 0 to 1. Using a combination of the other conditions gives a relationship between c and a:

$$am^2\left[E\left(\frac{1}{m}, 1 \to \lambda\right) + \left(1 - \frac{\lambda^2}{m^2}\right)\cdot F\left(\frac{1}{m}, 1 \to \lambda\right)\right] = -ic \tag{9}$$

The two parameters from the transformed problem, m and λ, are also linked, and further manipulation of the expressions gives

$$\lambda^2 = \frac{\displaystyle\int_1^m \frac{z^2 \cdot dz}{[(z^2 - m^2)\cdot(z^2 - 1)]^{\frac{1}{2}}}}{\displaystyle\int_1^m \frac{dz}{[(z^2 - m^2)\cdot(z^2 - 1)]^{\frac{1}{2}}}}. \tag{10}$$

This means that m and λ are dependent on each other, and so cannot be chosen separately. Because m is of more use I will choose a value of m and find λ from it. In order to choose a pair of values of c and d, and find the values of m and λ which correspond to them, it is necessary to use an iteration method to find the appropriate values of m.

These two integrals in Eqn (10) are standard elliptic integrals whose values can be found in tables, e.g. [5]. From these it is possible to find the values of the parameters m and λ from c and d. Some values are given in Table 1. For certain ranges of values, for example m close to 1 or m large, an approximation to the result in Eqn (10) can be used, based on expansions of the elliptic integrals.

Table 1
Some of the values of the parameters

m	λ	c	d	a
3	1.991	1	1.8	0.322
2	1.499	1	2.87	0.986
1.5	1.2499	1	4.926	2.654
1.2	1.1	1	10.969	8.328

When m is close to 1 the approximate result is

$$\lambda \simeq 1 + \frac{(m-1)}{2}. \tag{11}$$

When m is large

$$\lambda^2 \simeq \frac{m^2}{\ln(4m)}. \tag{12}$$

2.4. Using the transformation formula

Now that it is possible to find all the values we need in the transformation (3) we can proceed with using it. Hence we can choose any point z in the $z(x, y)$ plane. and Eqn (3) gives us the position of the corresponding point in the $Z(X, Y)$ plane. Since the problems are equivalent, we then can say that the pressure at a point $Z(X, Y)$ is the same as that at the corresponding point $z(x, y)$. Thus the problem is essentially solved.

We have the solution for the mixed boundary-value problem in the z-plane, and used it in the earlier paper [1] to evaluate the result at discrete points in an x, y-grid. In using the transformation (3) derived above we find the points in the Z-plane to which each point in the grid corresponds. This means we do not get a regular grid of points in the Z-plane, but have enough points to find the overall description of the pressure field. By choosing different points in the z-plane we can add detail to the most important region in the Z-plane.

The integrals are not straightforward, because the variable z' is complex, and the terms on the denominator of Eqn (3) tend to zero at certain points, when $z' = 1$ or when $z' = m$. This does not mean that the integral diverges, but it does mean that extra care is needed in carrying out the numerical integration. The techniques used are discussed more in the Ph.D. thesis by the author [6].

2.5. Pressure field

A result from the numerical integration of the transformation (3) is Fig. 3. We read in the result from the

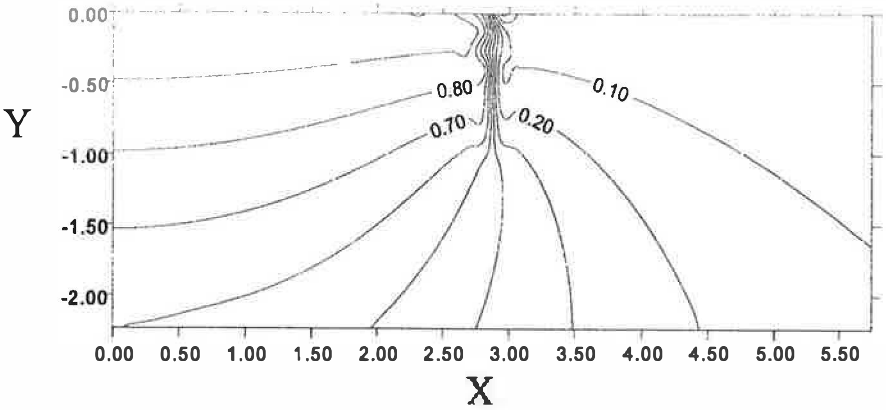

Fig. 3. Contours of pressure for the transformed mixed-boundary value problem.

mixed boundary value problem or 'flat' problem discussed earlier. Then the values of x and y from that problem are transformed to the equivalent points X and Y in the problem with depth, using Eqn (3). These are then output to a file, along with the corresponding pressure for the point. This information can then be plotted as a contour map of the pressure field. On a contour plot, flow occurs at right angles to the contour lines of constant pressure. The 'wobbly' contours near the wall are a function of the lack of detail in this region, and the discontinuity occurring there.

3. Flow rate

The value of the function U is transformed in the same way as the function P by the transformation $Z = f(z)$. In calculating the flow rate in the earlier part the only values of U which mattered were those where $y = 0$ and $x = -1$, and $y = 0$ and $x = +1$. These points map onto the points $X = -d$ and $X = +d$ on the X-axis of the transformed plane, which are the points which are needed to calculate flow in the Z-plane. Hence, since the same values apply at the 'key' points as for the untransformed problem, the flow, Q, is again given by

$$Q = \frac{k}{\mu} \cdot 2B \tag{13}$$

where B was defined in the previous paper [1] Eqn (9). B is a constant for any given problem in the 'flat', untransformed co-ordinates. This flow is the same as for the corresponding problem and means a given value of 'm' in the 'flat' problem corresponds to a specific ratio of c to d in the second problem. Some examples are given in Table 2. Note that when m is close to 1, the ratio c/d is equal to half of $m-1$. This is encouraging, as it suggests that a shallow cut into the ground has equal effect on the flow rate to a no-flow region on the surface of length equal to the length of both sides of that cut.

Table 2
Values of the different parameters as a function of m

m	d/c	c/d	B/P_0
1.001	2001.6	0.0005	2.8612
1.01	201	0.005	2.1296
1.1	20.99	0.048	1.4103
1.5	4.926	0.203	0.9503
2	2.87	0.348	0.7817
5	1.216	0.822	0.5261
10	0.837	1.195	0.4261
100	0.4	2.500	0.2622
1000	0.26	3.846	0.1894
10000	0.191	5.236	0.1482

4. Second conformal mapping problem

4.1. Problem definition

Now that we have the solution to a problem with thin footing walls it is natural to try to extend it to a more realistic problem, where the walls have thickness and depth. This then corresponds more closely to a real structure. The problem with thick walls is defined by Fig. 4. This can be related to the plane problem defined by Fig. 5. Note that the notation used here is slightly different from that used before, although this is not significant. Here the (a, b, c, d) replace the previous choice of (1, λ, m) from Fig. 2. This means that a different choice of arbitrary variables is used to the previous case, and the scale factor 'a' from the first example is chosen as 1. Because of this point on the z-plane nearest the origin, (a, 0), is not necessarily at the point (1, 0) as in the previous case.

The angles α_2 to α_9 are

$$\alpha_2 = \alpha_5 = \alpha_6 = \alpha_9 = \pi/2$$

$$\alpha_3 = \alpha_4 = \alpha_7 = \alpha_8 = 3\pi/2$$

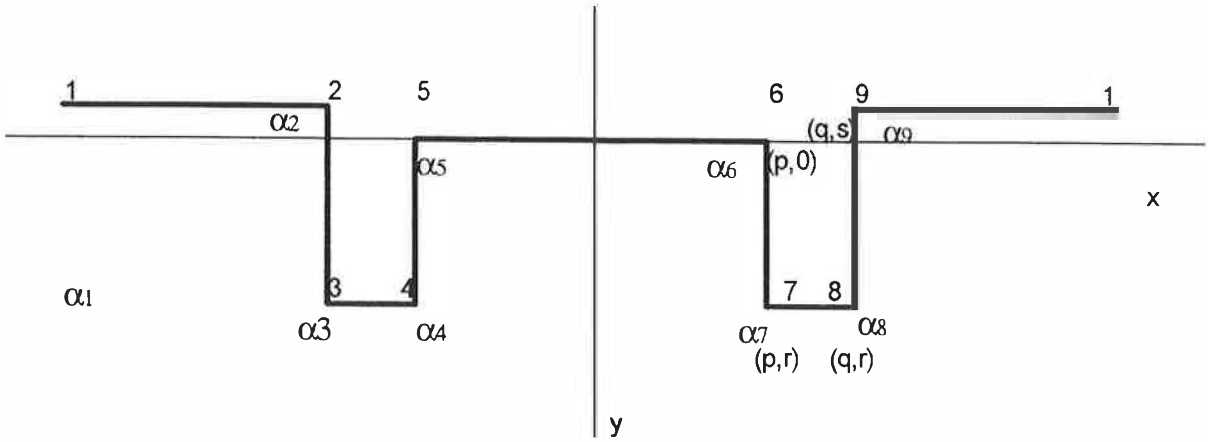

Fig. 4. Geometry of 'building'.

Fig. 5. The z-plane mapped from $z(x, y)$.

so that in the equation for the Schwartz–Christoffel transformation [4] we get

$$Z(X, Y) = f(z)$$

$$= A \int_0^z \frac{[(z'+b)(z'+c)(z'-c)(z'-b)]^{\frac{1}{2}}}{[(z'+a)(z'+d)(z'-d)(z'-a)]^{\frac{1}{2}}} dz' + b. \quad (14)$$

Then on setting b and z_0 equal to 0, and A to 1 as the arbitrary constants, then combining the terms we obtain

$$Z = f(z) = \int_0^z \frac{[(z'^2-c^2)(z'^2-b^2)]^{\frac{1}{2}}}{[(z'^2-d^2)(z'^2-a^2)]^{\frac{1}{2}}} dz'. \quad (15)$$

Equation(14) defines the transformation. It transforms the points $x = a, b, c, d$, in the $z(x, y)$ plane, to the points 6, 7, 8, 9 in the $Z(X, Y)$ plane. From the symmetry these four points also correspond to the points labelled 5, 4, 3, 2 in the X-negative half of the Z-plane.

Using the transformation, with the notation

$$f(x_1, y_1) = (X_1, Y_1) = X_1 + iY_1$$

at these known points we have

$$f(a, 0) = (p, 0) = p,$$

$$f(b, 0) = (p, r) = p + ir,$$

$$f(c, 0) = (q, r) = q + ir,$$

$$f(d, 0) = (q, s) = q + is,$$

and hence as before the result is defined. The first parameter p is given by

$$p = f(a, 0) = \int_0^a \frac{[(z'^2-c^2)(z'^2-b^2)]^{\frac{1}{2}}}{[(z'^2-d^2)(z'^2-a^2)]^{\frac{1}{2}}} dz'. \quad (16)$$

Similar expression can be found for q, r, and s. Integrating these expressions numerically for a range of values of a, b, c, d gives the results in the table below. It appears that these integral expressions can not be simplified further, and it is not possible to choose values of p, q, r and s and find, directly, the values of a, b, c and d which produce

Table 3
Values of the parameters from the second conformal mapping problem

a	b	c	d	p	q	r	s
1.00	1.200	1.800	2.00	1.323	1.624	−0.282	−0.001
1.00	1.400	1.600	2.00	1.445	1.477	−0.473	−0.003
1.00	1.010	1.990	2.00	1.032	1.963	−0.015	−0.000
1.00	1.0001	1.999	2.00	1.001	1.999	−0.000	−0.000
1.00	1.450	1.550	2.00	1.456	1.464	−0.498	−0.003
1.00	1.490	1.510	2.00	1.459	1.460	−0.508	−0.003
1.00	1.499	1.501	2.00	1.459	1.460	−0.509	−0.003
1.000	5.240	5.250	10.00	4.252	4.252	−5.083	−0.0066
1.000	2.000	8.900	10.00	2.609	7.359	−1.678	−0.0274
1.000	1.100	9.900	10.00	1.263	9.627	−0.159	−0.0027

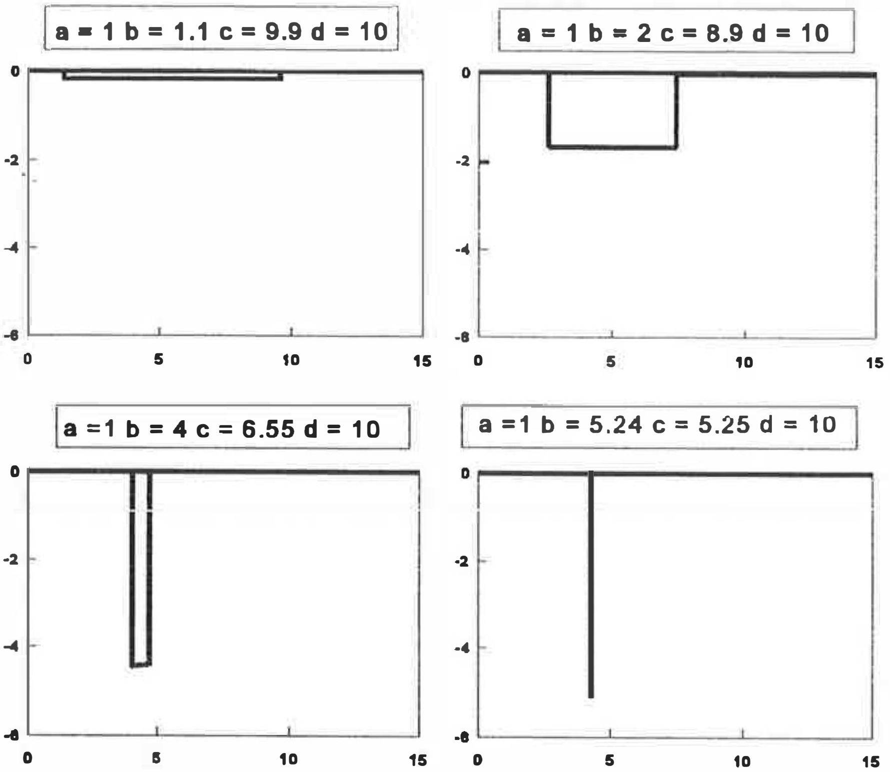

Fig. 6. Examples of equivalent shapes when $a = 1$ and $d = 10$.

them. However, it is fairly easy to find particular values of p, q, r and s by experimenting (trial and error) with the values of a, b, c and d within the integration program.

4.2. Flow rate

Using exactly the same analysis as for the flow rate of the first conformal mapping solution earlier we know that the flow rate is the same for any problem where the value of 'm' as used in chapter 4, (in this problem d is equivalent to 'm' if 'a' has the value 1).

In this case it means any system which has the same a and d has the same flow rate. Figure 6 shows equivalent shapes when $a = 1$ and $d = 10$.

4.3. Pressure field

By modifying the program used for the thin wall problem, the pressure field for the thick wall problem can be calculated using the transformation Eqn (14). The results for some of the set of solutions where a is 1 and d is 2 are shown in Figs 7–10. The first of these corresponds very

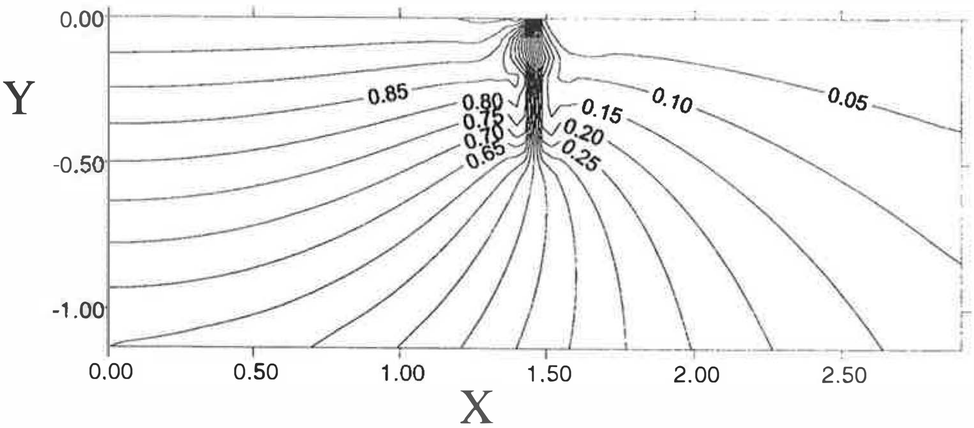

Fig. 7. Pressure field contours when $a = 1$, $b = 1.45$, $c = 1.55$, $d = 2$ (gives $p = 1.456$, $q = 1.464$, $r = -0.5$, $s = -0.04$).

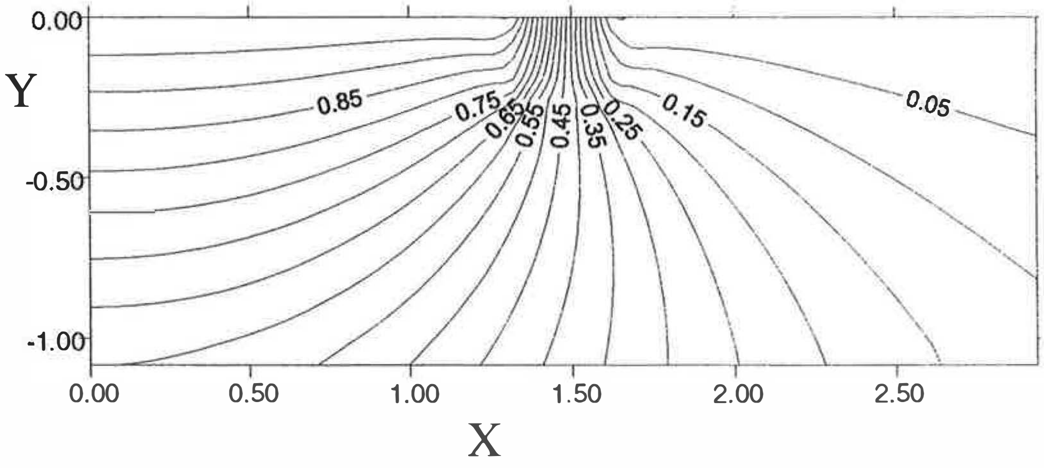

Fig. 8. Pressure field contours when $a = 1$, $b = 1.2$, $c = 1.8$, $d = 2$ (gives $p = 1.32$, $q = 1.62$, $r = -0.28$, $s = -0.001$).

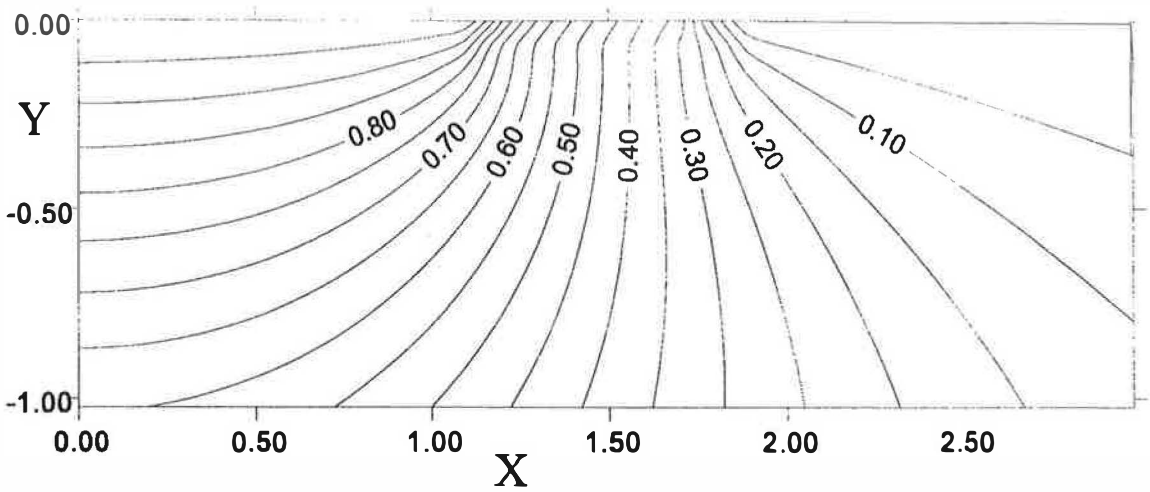

Fig. 9. Pressure field contours when $a = 1$, $b = 1.05$, $c = 1.95$, $d = 2$ (gives $p = 1.12$, $q = 1.86$, $r = -0.08$, $s = -0.001$).

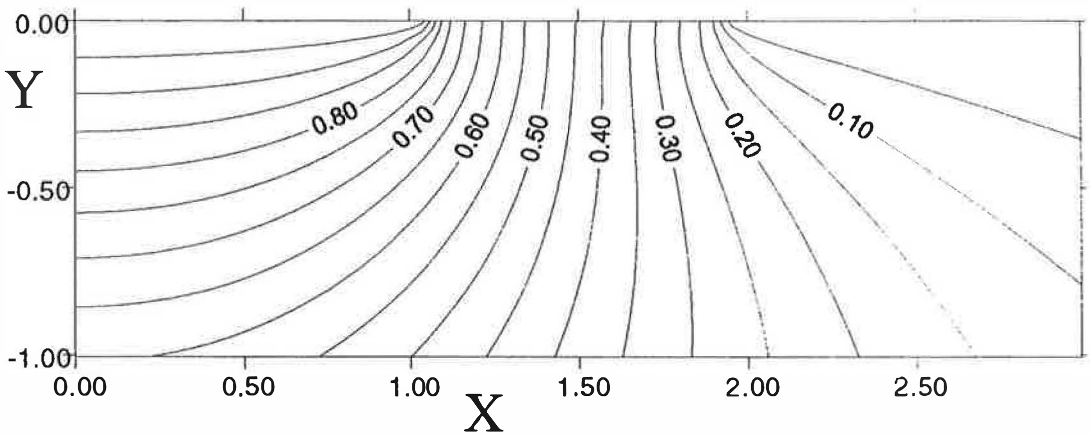

Fig. 10. Pressure field contours when $a = 1$, $b = 1.01$, $c = 1.99$, $d = 2$ (gives $p = 1.03$, $q = 1.96$, $r = -0.16$, $s = 0$).

closely to that of Fig. 3 shown earlier. The others show the result for progressively 'fatter' but less deep walls, leading to Fig. 10, which is close to the result for the untransformed problem of Fig. 2. Note that the model provides no data at all for the region within the wall, so any lines within the walls are a product of the plotting programme.

5. Conclusions

In this paper the solution to a mixed value problem from a previous paper has been transformed onto two different geometries. The relationship between the flow rates for each of these soil gas problems, and the associated pressure fields have been found. This gives a useful technique for finding solutions to problems which are difficult to find from other methods.

A third paper follows which describes an experiment designed to test these results, and the comparison with another experimental technique. This technique could be extended to other problems, and applied to other types of flow besides soil gas, with heat loss through solid ground floors being the most obvious.

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