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Summary The paper outlines a derivation of a fundamental equation of humidity for any enclosed space or set of spaces exchanging air with each other and with the external environment. The solution enables the humidity levels to be predicted for any arbitrarily changing set of conditions. A number of simpler solutions for room humidity levels are given and discussed. A program to allow users to evaluate different hypothetical effects in the general case easily is cited. A brief discussion, using some specific examples, is made of the various factors contributing to the resultant room humidity and the effectiveness of ventilation. The analytical form of the solution given to the equations allows a user to assemble a model of the humidity response of a building to any degree of complexity using simple calculations.

Humidity levels in buildings with variable moisture sources, ventilation rates and environmental conditions

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List of symbols

$r(t)$	Moisture production rate at time t within a room(suffix j for the jth room in a building)(kg s^{-1})
$f(t)$	Forced and infiltration ventilation rate of air at external humidity condition at time t (m^3 s^{-1})
$\phi_r(t)$ or ϕ_r	Room relative humidity at time t (%)
$\phi_a(t)$ or ϕ_a	External relative humidity at time t (%)
$\theta_r(t)$ or θ_r	Room temperature at time t (°C)
$\theta_a(t)$ or θ_a	External temperature at time t (°C)
$g_{sr}(\theta)$	Saturation moisture content of room air at temperature θ (kg/kg dry air)
$g_{sa}(\theta)$	Saturation moisture content of external air at temperature θ (kg/kg dry air) (Note: The functions g_{sr}, g_{sa} may also be written as time-dependent time since θ is time-dependent.)
p_{sr}	Saturation vapour pressure of room at time t (Pa)
ρ	Density of air (kg m^{-3})
V	Room volume (m^3)
τ	Elapsed time from reference time zero (s)
ϕ_i	Initial room humidity at reference time zero (s)
κ	Rate of (linear) change of room temperature change (K s^{-1})
X,Y,Z	Coefficients of approximation for $g_{sr}(\theta) = X\theta^2+Y\theta+Z$; $X=1.67935 \times 10^{-5}$, $Y = 0.000260944$, $Z=0.00332212$
j	Room suffix (taking values 0 ... n) for an n-roomed building; external environment denoted by suffix 0
$S(\theta)$	Defined in text
$B(t)$	Defined in text
$C(t)$	Defined in text
$\Lambda(t)$	Defined in text

1 Introduction

There is increasing emphasis on producing buildings so well sealed that fortuitous ventilation is minimised[1]. This means that deliberate mechanical ventilation and controls are often essential to prevent damaging condensation in areas of high humidity. Experience clearly shows that over-designing such systems will only lead the user to repudiate their use totally. This is due to the cost associated with excessive ventilation energy loss and to the resulting comfort effects[2]. However no simple, effective algorithms are readily available to allow an engineer to match ventilation patterns and rates with likely variable patterns of use[3]. Expensive and complex simulation packages must be invoked. A model is presented here, with a simple procedure capable of treating the most complex changing patterns of ventilation rate and other environmental conditions. The solution is suitable for engineering calculation or implementation on a spreadsheet by providing a simple closed form of solution to the ventilation model equations.

Steady-state evaluations using mean values are usually unsatisfactory since they tend to over-estimate required trickle rates and conversely under-estimate the boost rates required to prevent surface condensation. These effects arise mainly because in the steady state the capacity of the room and building to absorb moisture is neglected. Conversely the inability of the room to accept moisture during those parts of the daily cycle when its air temperature is falling is a further source of error. Ventilation always imposes some net energy loss on a building. A designer may wish to incur a minimum safe energy loss penalty arising from ventilation without involving the complication of heat exchangers. Transient conditions can then lead to important cumulative effects, which should then be included in any assessment of the minimum acceptable ventilation rates.

At present an engineer has few rigorous practical techniques for sizing the ventilation system according to the dynamics of the building conditions. Parameters include the background and the boost levels, control point humidities and comfort levels in a room under both winter and summer conditions. This paper presents a solution to the dynamic problem for a room in a readily calculable form which can be implemented through a computer spreadsheet.

2 Formulation of basic problem

First we consider a single room or space containing moisture sources which may be varying in time. The room exchanges ventilation air at a variable rate (trickle, boost etc.) with the external environment, whose temperature and humidity may also vary in time. Clearly the room temperature and internal humidity may also be expected to vary with time.

Moisture produced within the room is added to that introduced through ventilation air from outside. This input is offset by moisture lost in the extracted ventilation air, but there is generally a change in the moisture content and hence humidity in the room as a whole, if good mixing can be assumed. If this is not the case then the room may have to be zoned. (See below for discussion of assemblies of zones.)

Summing the internally produced moisture and the moisture introduced with the ventilation air gives for total moisture gain rate in the room (kg s^{-1}):

$$r(t) + f(t)\rho\phi_a(t)g_{sa}(\theta_a)/100 \quad (1)$$

given the approximation that the percentage saturation is not significantly different from the relative humidity. This is generally acceptable over the range ambient conditions in inhabited buildings.

Similarly the total moisture loss rate (kg s^{-1}) is given by:

$$f(t)\rho\phi_r(t)g_{sr}(\theta_r)/100 \quad (2)$$

The net rate of change of moisture per unit mass is given by:

$$d(\phi g_{sr})/100dt = (d\phi_r/100dt)g_{sr} + \phi_r(dg_{sr}/100dt) \quad (3)$$

where the time derivative of g_{sr} is defined by:

$$dg_{sr}/dt = (dg_{sr}/dp_{sr})(dp_{sr}/dT_r)(dT_r/dt) \quad (4)$$

The first two terms in equation 4 are determined by the properties of mixtures of water vapour and air at normal temperatures and pressures; they could be evaluated using equations C1.13 and C1.9 respectively in *CIBSE Guide* C[4] defining g_{sr} in terms of p_{sr} and p_{sr} in terms of the absolute air temperature T_r. However, these complex expressions may be approximated adequately (Appendix 1) by a second-order polynomial for gsr as a function of the air dry bulb temperature θ_r. This function is differentiated to give equation 4 directly. The function dg_{sr}/dt is defined completely by the user and contains no unknown quantities. Thus the simplified form for dg_{sr}/dt is:

$$\begin{aligned} dg_{sr}/dt &= (dg_{sr}/d\theta_r)(d\theta_r/dt) \\ &= (d/d\theta_r)(X\theta_r^2 + Y\theta_r + Z)(d\theta_r/dt) \\ &= (2X\theta_r(t) + Y)(d\theta_r/dt) = S(\theta_r)(d\theta_r/dt) \end{aligned} \quad (5)$$

where the expression for $dg_{sr}/d\theta$ in terms of X and Y is written as $S(\theta_r)$. Hence the final equation describing the relation between the controlling parameters of the room relative humidity is found by equating the difference of net moisture gain (or loss) from the room given respectively by equations 1 and 2 to the rate of change given by equations 3 and 5. The former equations relate to the whole room and the latter to a unit mass of air within the room; there is a scaling factor of ρV between them. Thus:

$$\begin{aligned} &100r(t)/\rho Vg_{sr}(t) + f(t)\phi_a(t)g_{sa}(t)/Vg_{sr}(t) \\ &- [f(t)/V + (S(\theta_r)/g_{sr}(t))(d\theta_r/dt)]\phi_r = d\phi_r/dt \end{aligned} \quad (6)$$

This is a general linear first-order differential soluble by an integrating factor[5]. It may be written more concisely by grouping the known functions and values into single expressions. Hence:

$$B(t) = 100r(t)/\rho Vg_{sr}(t) + f(t)\phi_a g_{sa}(t)/Vg_{sr}(t) \quad (7)$$

and similarly:

$$C(t) = f(t)/V + (S(\theta_r)/g_{sr}(t))(d\theta_r/dt) \quad (8)$$

Thus equation 6 may be written in terms of the functions $B(t)$ and $C(t)$ which are determined by the various factors affecting the room humidity. They are therefore readily calculable to give:

$$d\phi_r/dt + C(t)\phi_r = B(t) \quad (9)$$

The room humidity at any time t subject to the combination of influences described above is given by solving equation 9 with an integrating factor:

$$\phi_r(\tau) = \exp(-\Lambda(\tau))(\int_0^t B(t)\exp(\Lambda(t))dt + \phi_i) \quad (10)$$

This a closed form of the humidity at time τ within any uniform space as a solution of equation 6. The exponent Λ in equation 10 is calculated from

$$\Lambda(\tau) = \int_0^t C(t)dt \quad (11)$$

The humidity is consequently determined by summing the exponential weights of changing environmental factors in $B(t)$ and $C(t)$. The formal solution may appear intractable but it is a simple integral patient of numerical solution. A spreadsheet, for example, can readily be used to determine values of the room humidity ϕ_r. The formal solution assumes the analyticity of the various functions, but in practice they may contain discontinuities with sudden changes in moisture production or temperature. This can be addressed by describing the discontinuous functions by Fourier series over a defined range; the series are analytic. Conventional treatments[6–8] do not allow for time dependency. A complete review is not appropriate here, but see References 8 and 9 for typical comparative treatments and their solutions. This treatment continues with cases where one or more of the environmental influences remain relatively constant. This is of limited applicability but illustrates some features common to more complex situations examined later.

2.1 Constant external environmental conditions, internal temperature and ventilation rate

Although this case may appear unrealistic, it can give an immediate estimate of the response rate for a given fan rating in clearing a given initial humidity from a room. The ratio V/f has units of time, identifying it with the system time constant. f/V is the air change rate per second. Integrating equations 10 and 11 gives the room humidity at time t:

$$\begin{aligned} \phi_r(\tau) = &(100r/\rho g_{sr}f + \phi_a g_{sa}/g_{sr})(1 - \exp(-f\tau/V)) \\ &+ \phi_i\exp(-f\tau/V) \end{aligned} \quad (12)$$

Figures 1 and 2 shows the result of plotting ϕ_r as a function of time at various values of f/V for a 'low' (5.8×10^{-5} kg s^{-1}) and a 'high' (1.6×10^{-4} kg s^{-1}) moisture production rate (defined according to Table 3 of *BS 5250:1989*)[10]. Ventilation begins suddenly in a room initially at 55% relative humidity.

The curves show that with f/V ratios below around 3×10^{-4} s^{-1}(approx 1ac h^{-1}) the ventilation is ineffective. Humidity only decreases over periods comparable with the usual moisture production times. The data shows that f/V ratios of up to 1×10^{-3} s^{-1} (approx 3 ac h^{-1}) are necessary to produce response times compatible with the acceptable humidity limits within rooms. Thus, for example, taking the usual moisture production rates described in *BS 5250*[10], Figures 1 and 2 suggest that f/V ratios of around 3–5 $\times 10^{-3}$ s^{-1} (approx 9–16 ac h^{-1}) are required for a ventilation system in boost mode if its response is to be comparable with that necessary for comfort and condensation avoidance.

Surveyor
10 April 1997

CONTRACTS

● Oscar Faber has been commissioned to assess the potential for new stations, station improvements and additional services on the rail line between Birmingham New Street and Nuneaton in Warwickshire. A major aim of the study is to develop proposals from a number of previous rail line and station studies to see if an operating surplus can be achieved when they are considered as a package, rather than in isolation. The studies had identified a number of common features relating to operating and capital costs, such as rolling stock leasing charges and the cost of disabled access at stations.

Rail Privatisation News
3 April 1997

Nuneaton line study

Oscar Faber is to assess the potential for new stations, station improvements and additional services on the Birmingham – Nuneaton route. The study has been commissioned by a consortium of Warwickshire County Council, North Warwickshire Borough Council, Nuneaton & Bedworth Borough Council, West Midlands Passenger Transport Executive and Arley Parish Council.

The findings of several existing studies will be developed as a package for the line that would hopefully generate an operating surplus when factors such as train leasing charges are taken into account. In particular, Oscar Faber will examine the service options for a new station at Arley and the potential for a station at Galley Common/Grove Farm near Nuneaton.

Highways
March 1997

Oscar Faber has been commissioned by Coventry City Council to review the 1990 Coventry Integrated Transport study and project forecasts further forward to the year 2001. The firm has also recently won a commission from the ExpressRoute Consortium, currently bidding for the Weald and Downland DBFO in Kent, to advise on all

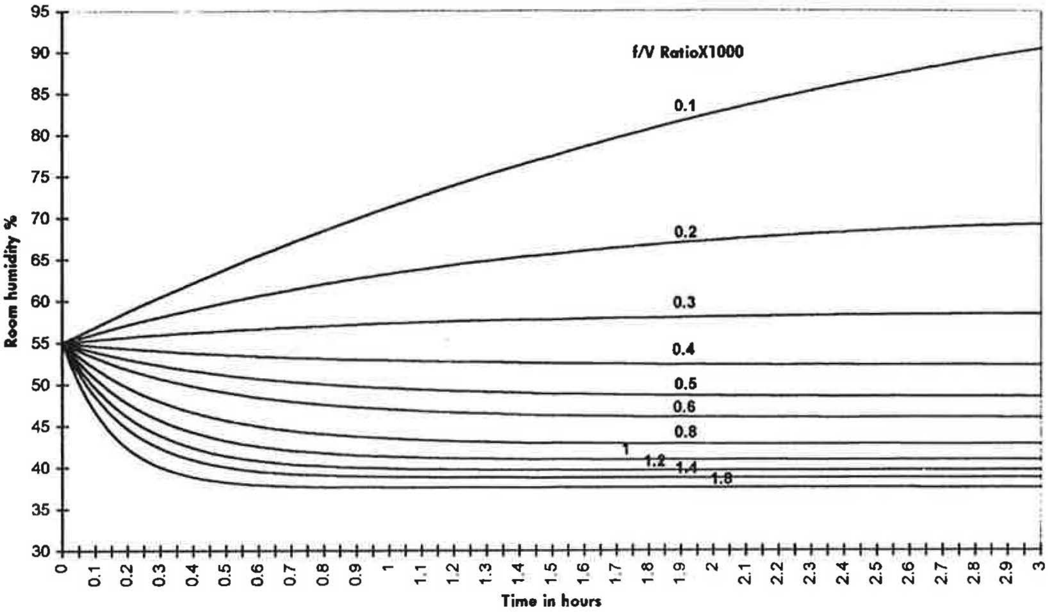

Figure 1 Room humidity versus time for various f/V ratios (equation 10) under dry occupancy (BS 5250 Table 3; External conditions 5°C, 80%RH; Internal temperature 18°C)

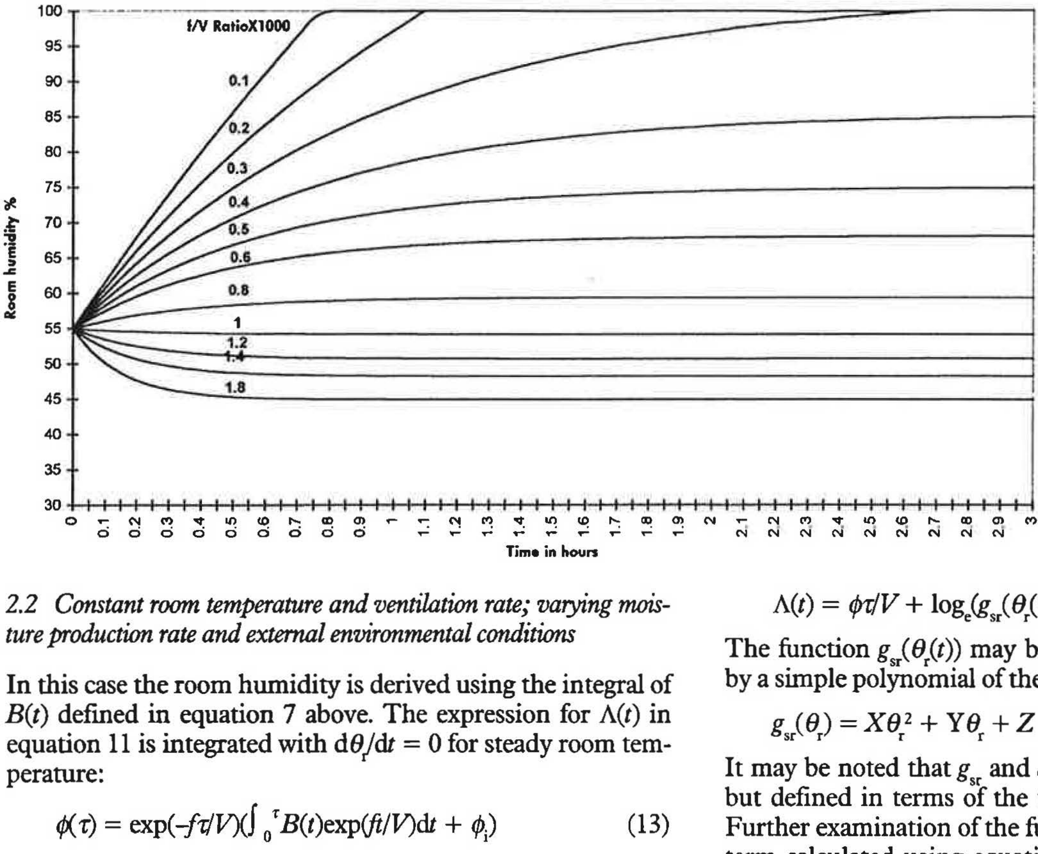

Figure 2 Room humidity versus time for various f/V ratios (equation 10) under wet occupancy (BS 5250 Table 3; External conditions 5°C, 80%RH; Internal temperature 18°C)

2.2 Constant room temperature and ventilation rate; varying moisture production rate and external environmental conditions

In this case the room humidity is derived using the integral of $B(t)$ defined in equation 7 above. The expression for $\Lambda(t)$ in equation 11 is integrated with $d\theta_r/dt = 0$ for steady room temperature:

$$\phi(\tau) = \exp(-f\tau/V)(\int_0^\tau B(t)\exp(ft/V)dt + \phi_i) \qquad (13)$$

The exponential terms can be evaluated directly. The integral must be found numerically using the Simpson's or trapezoidal rule[11] and the expression for $B(t)$ given in equation 7. This can easily be done with a spreadsheet. Each disturbance in the moisture or environmental conditions grows at the exponential rate f/V and correspondingly decays at the same rate. Figures 3 and 4 show the results of evaluating equation 13 for the simple moisture production rates. Figure 5 shows the varying moisture production rate. Figures 6–8 show the effect of adding temperature variation.

2.3 Discrete ventilation rates with all other parameters varying

The room temperature rises or falls at a defined linear rate k (K s^{-1}). After some manipulation (see Appendix 1) the integral for $\Lambda(t)$ gives

$$\Lambda(t) = \phi\tau/V + \log_e(g_{sr}(\theta_r(\tau))/g_{sr}(\theta_r(0))) \qquad (14)$$

The function $g_{sr}(\theta_r(t))$ may be reasonably accurately replaced by a simple polynomial of the form (Appendix 1):

$$g_{sr}(\theta_r) = X\theta_r^2 + Y\theta_r + Z \qquad (15)$$

It may be noted that g_{sr} and S are treated as time-dependent, but defined in terms of the time varying room temperature. Further examination of the function 14 shows that the second term calculated using equation 15 (see Appendix 1) may be approximated by a simple linear expression over the range –2–22°C when the room temperature moves linearly up or down. Figure 9 plots equation 14 as a function of θ. This gives for the final value for the exponent

$$\Lambda(t) = (f/V + 0.033k)t \qquad (16)$$

Thus a falling or rising room temperature is equivalent to an apparent change in fan output, in this case of $0.033k$, where k is positive for a rising temperature and vice versa. Where values of f/V and $0.033k$ are comparable the effects of the latter term may be considerable. For example, with a system working with $f/V = 0.83 \times 10^{-3}$ (3 ac h^{-1}) then a temperature drop at 0.5 K min^{-1} reduces the effective air change rate by 33%.

When a room is ventilated mechanically, the ventilation rate generally takes only one of two or three discrete values. At

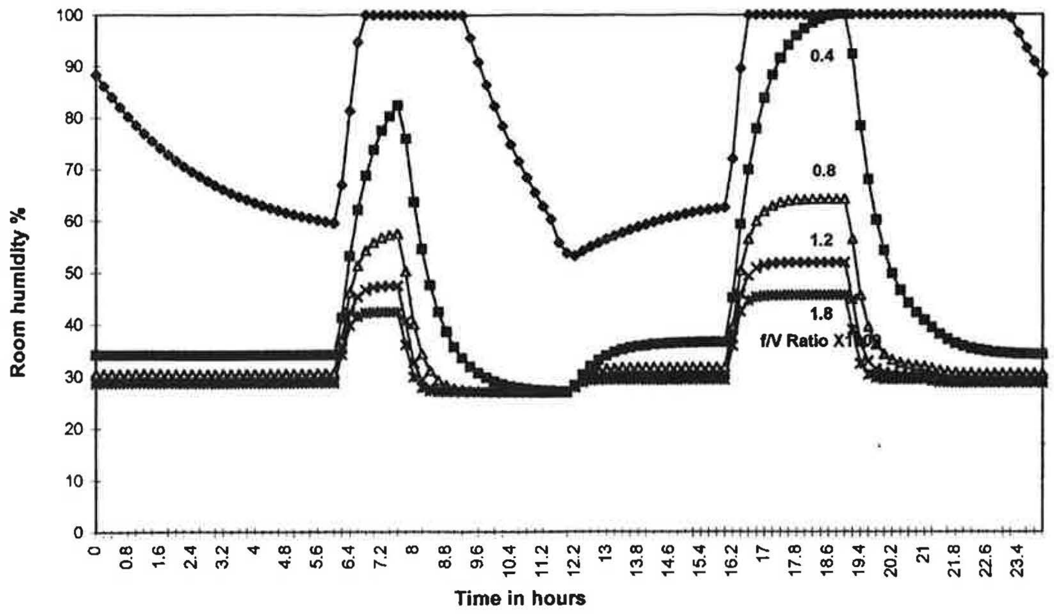

Figure 3 Room humidity versus time for the variable source shown in Figure 5 (equation 10) (sections through Figure 4)

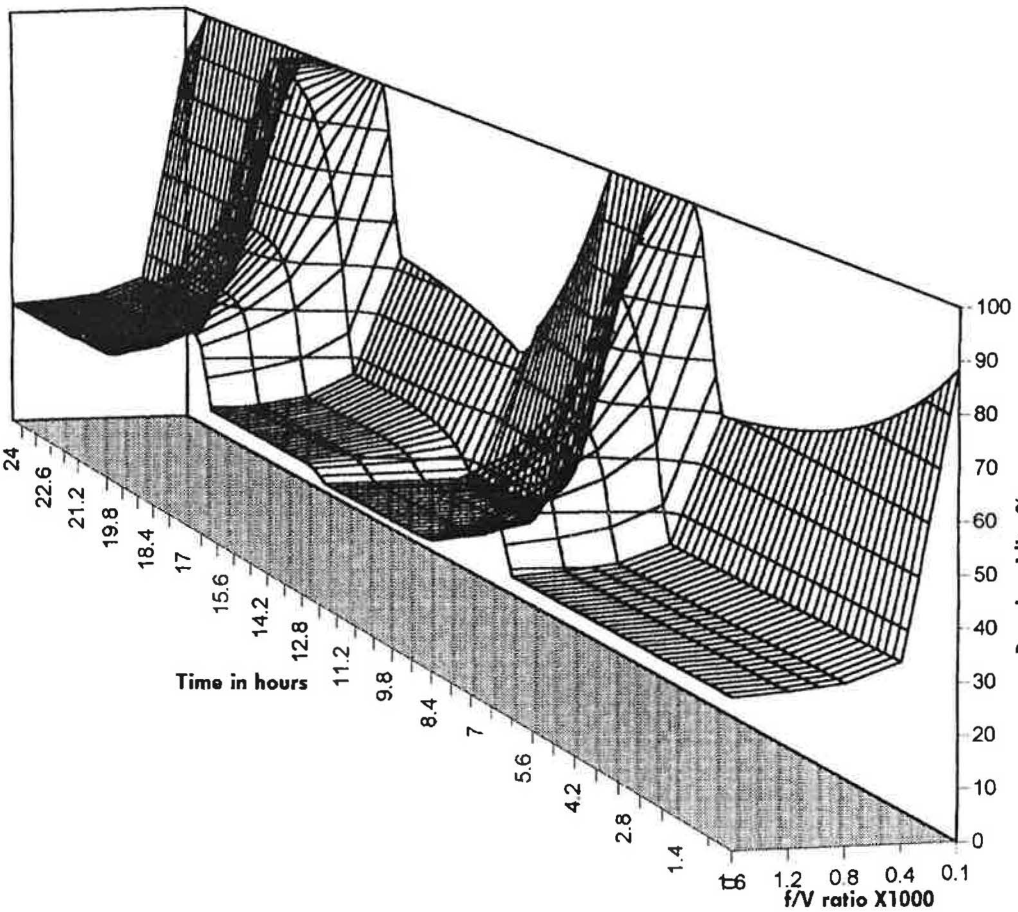

Figure 4 Functional surface of room humidity, f/V ratio, time and moisture rates from Figure 5 (equation 10)

some fan power setting the rate is sensibly constant. The ventilation rate then takes up a series of level values (designated as f_i, where $i = 1,2$ etc.) for varying periods of time (designated as τ_i, where $i = 1,2$ etc.) These values would include natural ventilation or infiltration. This situation obtains either under user control or automatically. (See section 3.)

3 Evaluating and extending the method

Having examined some of the simpler analytical cases, we may now look at cases where all the parameters affecting the resulting room humidity are varying. These results are of course limited to a single room exchanging air with the outside environment. The algorithm is equally valid for an assembly of rooms exchanging air with each other and with the external environment, by simply replacing the outside air terms with a summation of the values appropriate to the other rooms (and including the outside environment values as well).This more computationally complex case is addressed briefly below, but the details of such a whole-building simulation will be published separately. All the previous analysis will hold with $B(t)$ and $C(t)$ defined by:

Figure 5 Moisture production rate versus time for results in Figures 3, 4, 6 and 8)

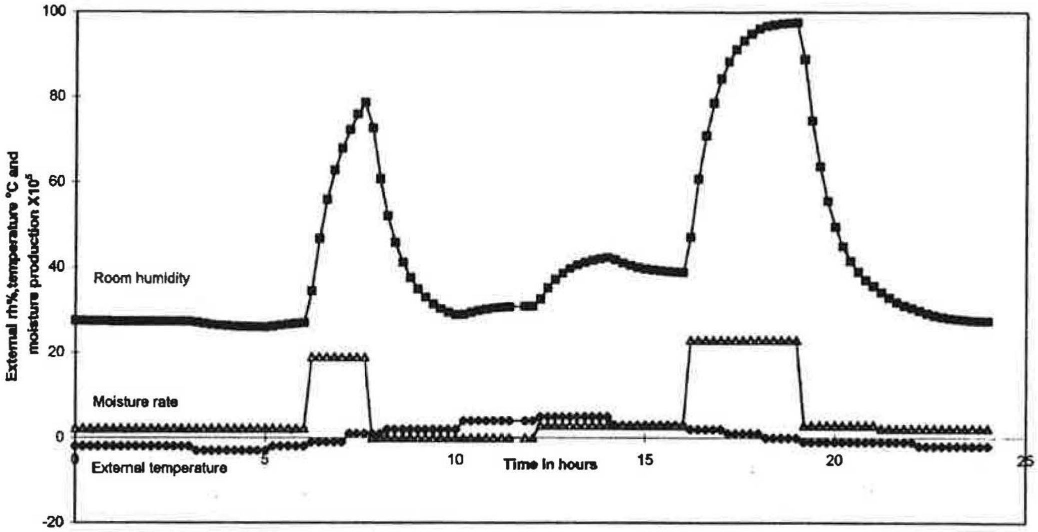

Figure 6 Comparative plots of room humidity, external temperature and moisture production for $f/V=0.4\times10^{-3}$

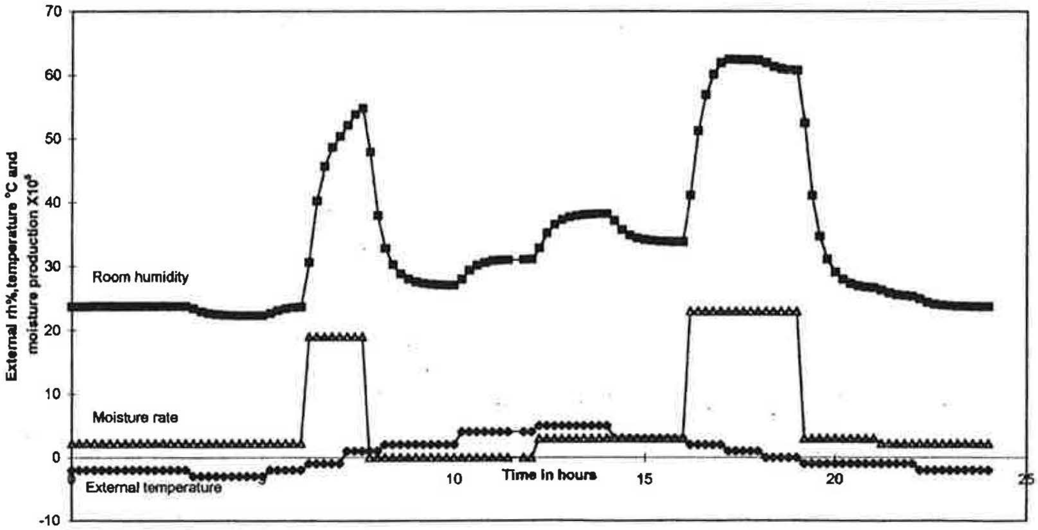

Figure 7 Comparative plots of room humidity, external temperature and moisture production for $f/V=0.8\times10^{-3}$

$$B(t) = 100r(t)/\rho V g_{sr}(t) + \Sigma_{j=0}{}^{n}\left(f_j(t)\phi_j(t)g_{sj}(t)/Vg_{sr}(t)\right) \quad (17)$$

and

$$C(t) = \Sigma_{j=0}{}^{n}\left(f_j(t)/V\right) + \left(S(\theta_r)/g_{sr}(t)\right)/(\mathrm{d}\theta_r/\mathrm{d}t) \quad (18)$$

where the suffix j refers to each space in the building (generally a single closed space, but large rooms may be subdivided into separate zones), including the external environment that

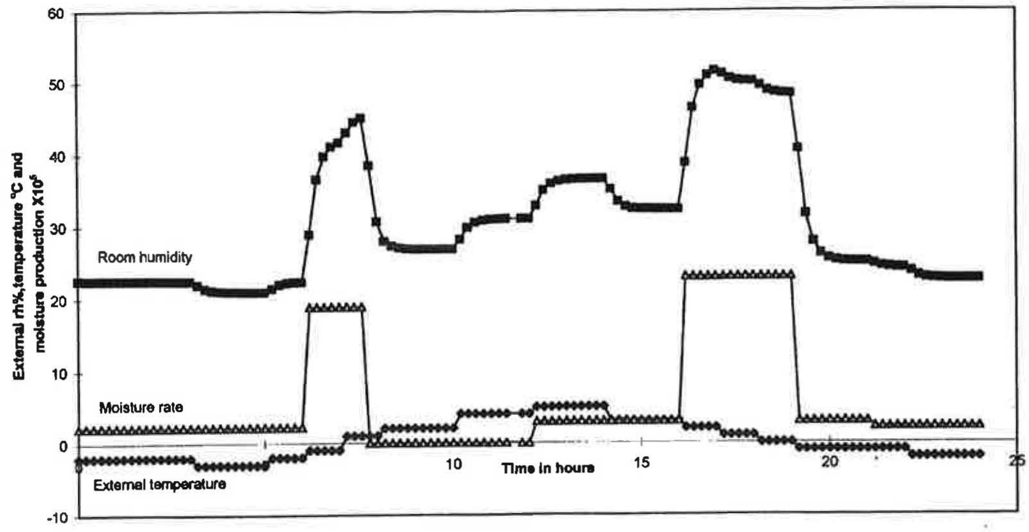

Figure 8 Comparative plots of room humidity, external temperature and moisture production for $f/V = 1.2 \times 10^{-3}$

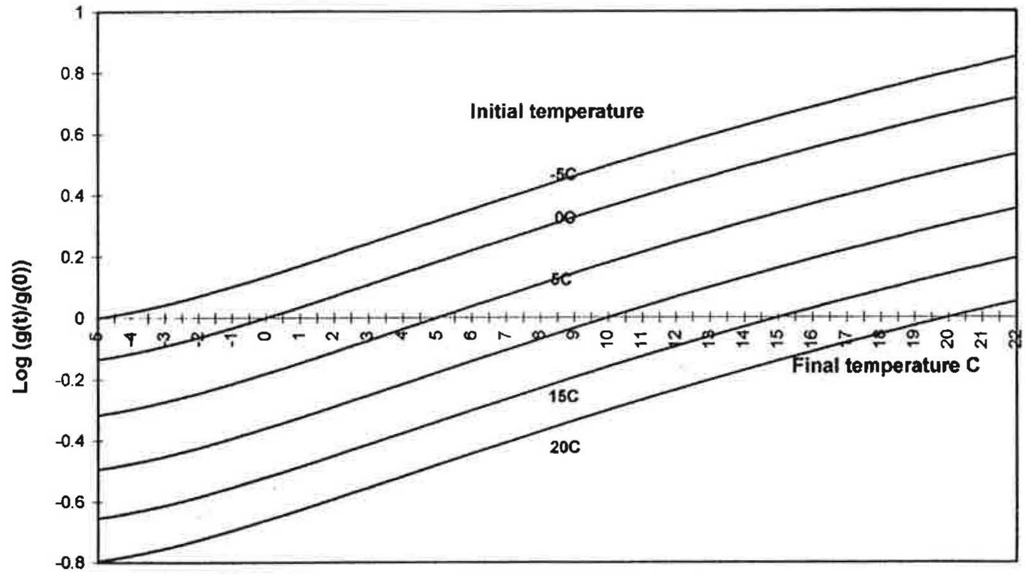

Figure 9 Plot of log(gsr(t)/gsr(0)) versus final room temperature for various initial values of room temperature

the room may be exchanging air with. The functions $S(\theta_r)$, $r(t)$ and $g_{sr}(t)$ refer as before to the room itself.

3.1 Numerical evaluation of solutions

Equations 10 and 11 are simply evaluated in the single-room case in a spreadsheet or similar package, where the integral terms are replaced by simple trapezoidal sums[11]. Reference 3 is a simple macro in Visual BASIC® for use with Microsoft Excel® 5.0[13]. The cell definition names are listed in Appendix 3. The algorithm evaluates the values of equations 10 and 11 for each time step. Due to the exponential nature of the solutions to equations 10 and 11, the time steps (of the order of a few seconds) must be relatively small, otherwise serious errors will grow in the developing values of the room humidity, thus damaging the accuracy of the predictions. Not all the interim step results will therefore be required, and need only be displayed after suitable blocks of ten or more steps. A further caution is necessary if very large values of f/V are to be used ($>15 \times 10{-}3$ s^{-1} or 50 ac h^{-1} approx.) since then intermediate values in the computation may cause overflow in single-precision arithmetic.

3 Conclusions

The fundamental room humidity equations 10 and 11 enable the user to determine the humidity level after any interval of time from any initial humidity condition, subject to arbitrarily chosen changing environmental conditions. The equations may be used iteratively to determine the humidity levels within a building treated as an assembly of rooms, or zones within a room. The equation is readily evaluated using a simple program. As such it may be used as the basis of a simulation package when linked with air movement and heat production simulation.

The solution demonstrates that the key parameters in determining the effectiveness and speed of response are the sum of the ratio f/V (air change rate) of the system including infiltration, the rate of change of temperature over the interval under consideration, and the functions $B(t)$ and $C(t)$ defined in equations 7 and 8. The functions $B(t)$ and $C(t)$ have the units of s-1 and therefore may be identified as reciprocal system time constants.

5 Acknowledgements

The author acknowledges permission to publish this paper from the Director of the British Board of Agrément, Professor P C Hewlett.

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Appendix 1: Simple second-order functional fit to the saturation moisture content of air

The saturation moisture content of air as a function of the air dry bulb temperature in degrees Celsius may be approximated by:

$$g_{sr}(\theta) = 1.67935 \times 10^{-5}\theta^2 + 0.000260944\theta + 0.00332212 \quad (19)$$

This function is derived by forming the second-order least-squares fit at one degree intervals (--10°C to 30°C) to the values of saturation moisture content given in *CIBSE Guide C*. Over the range θ=–5°C to 25°C this function has a maximum estimated error of less than 4 %. Clearly if θ_r is time-dependent then so is gsr.

Similarly the derivative of g_{sr} with respect to θ defined by $S(\theta)$ may be found. When θ is a rising or falling linear function of time, as discussed in section 2.3, and the fan rate is sensibly constant, then we have to evaluate:

$$\Lambda(\tau) = \int_0^{\tau} C(t)\mathrm{d}t = \int_0^{\tau}(f/V)\mathrm{d}t + \int_0^{\tau}(S(\theta_r)/g_{sr}(\theta_r)(\mathrm{d}\theta_r/\mathrm{d}t)\mathrm{d}t \quad (20)$$

The first integral is obtained as a constant $f\tau/V$. For the second integral, using $\theta_r(t) = \theta_0 + kt$ and $S(\theta_r) = \mathrm{d}g_{sr}/\mathrm{d}\theta_r$, then substituting in the second integral and changing the variable from t to θ_r gives:

$$\int_0^{\tau}(S(\theta_r)/g_{sr}(\theta_r)(\mathrm{d}\theta_r/\mathrm{d}t)\mathrm{d}t = \int_{\theta(0)}^{\theta(\tau)}(S(\theta_r)/g_{sr}(\theta_r))\mathrm{d}\theta_r = \log(g_{sr}(\theta_r(t))/g_{sr}(\theta_r(0))) \quad (21)$$

If the value of the last term of equation 21 is plotted for a range of $\theta_r(0)$ and $\theta_r(t)$ (Figure 9), then equation 21 may be approximated by the simple linear expression $0.033(\theta_r(t) - \theta_r(0)) = 0.033kt$. (Correlation coefficient = 0.99.)

Appendix 2: Spreadsheet cell name definitions for use with Reference 12

WORKSHEET NAME	DEFINITION
volume	Room volume m^3
rhint	Initial room humidity %
timehrs	Time in hours (first table entry)
moisture	Moisture production rate kg s^{-1} (first table entry)
tmpin	Internal dry bulb temperature °C(first table entry)
rhext	External humidity % (first table entry)
tmpex	External dry bulb temperature °C(first table entry)
ventr	Ventilation rate litre s^{-1} (first table entry)

VENTILATED ROOM HUMIDITY SIMULATION: Sample Table Data Entry

JOB No.	

Room Volume (m^3)

Internal Conditions			External Conditions		Ventilation rate (l s^{-1})	Resultant Humidity %
Time (hrs)	Moisture Rate (kg s^{-1})	Temp. (°C)	Humidity (%)	Temp. (°C)		