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Optimal Control of a Single Zone Environmental Space

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The problem of finding an optimal control scheme for space heating is explored. A bilinear model for the plant is linearized, and an optimal control law is found by solving the Riccati equation. The output response of the system to impulse and step disturbances is investigated. A simplified design leading to a suboptimal control law, where only the dominant states of the plant are considered is given. The system performance with optimal and suboptimal control is compared. A reduced order estimator is designed and system performance with the estimator is tested under simulated step disturbances. From the view point of zone air temperature control, the system performance with suboptimal controller is shown to be acceptable.

NOMENCLATURE

- $\mathbf{A}$ matrix
- A surface area
- A_{d1} surface area of first node
- A_w window area
- a_s storage tank heat loss coefficient
- a_{sb} heat transfer coefficient
- a_z zone heat loss coefficient
- $a_{11}, a_{12}, a_{21}, a_{22}$ matrix elements
- $\mathbf{B}$ matrix
- b_2 matrix element
- C_s thermal capacity of the storage tank
- C_z zone air mass thermal capacity
- COP coefficient of performance
- COP_{max} maximum COP
- C_{d1} through C_{d5} thermal capacity of nodes 1 through 5
- C_9 through C_{22} coefficients computed from system parameters
- $\mathbf{E}$ matrix
- e output error
- e_1 element of matrix $\mathbf{E}$
- f observer characteristic function
- g_1, g_2 scalars in observer model
- $\mathbf{H}$ output matrix
- h heat transfer coefficient
- I internal heat sources
- ΔI variations in I from the operating condition
- J cost functional
- $\mathbf{K}, \mathbf{K}_1$ gain vector
- K_1, K_2, K_3 feedback gains
- L observer gain
- $\mathbf{Q}$ weighting matrix on state variables
- $\mathbf{Q}_1, \mathbf{Q}_2$ weighting matrices
- R weighting on control input
- S matrix variable in Riccati equation
- S_g solar gains
- ΔS_g variations in S_g from operating point
- $\mathbf{T}$ temperature vector
- $\mathbf{T}_0$ temperature vector at the operating point
- $\Delta\mathbf{T}$ variations in $\mathbf{T}$ about $\mathbf{T}_0$
- T_s storage temperature
- T_o source heat temperature to the heat pump
- T_{d1} through T_{d5} nodal temperatures of the mass wall
- ΔT_{max} maximum temperature differential for the heat pump
- T_z zone air temperature
- ΔT_s variations in T_s about the operating point T_{s0}
- ΔT_z variations in T_z about the operating point T_{z0}
- T_a ambient temperature
- ΔT_a variations in T_a about operating point
- t time
- U_{1max} capacity of the circulating pump
- U_2 control variable, normalized input energy to the heat pump
- U_{2max} rated input power to the heat pump
- ΔU_2 variations in U_2 about U_{20}
- $\Delta\mathbf{W}$ disturbance vector
- ΔX distance between nodes
- y output (zone temperature)
- y_d desired output (setpoint)
- Z variable, observer model
- ΔZ variations in Z

Greek letters

- ξ state variable for the output integrator
- $\Delta\xi$ variations in ξ
- α absorptivity
- τ_s storage time constant
- τ_d mass wall time constant
- τ_z zone air mass time constant

Superscripts

- $\dot{}$ derivative with respect to time
- $\hat{}$ estimated value
- $*$ optimal value
- $'$ transpose

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1. INTRODUCTION

THE TASK of maintaining an environmental space or zone within thermal comfort temperature limits, as required by the occupants of a space, is one of the most important control problems of practical interest. Since, either central or unitary heating, ventilating and air conditioning (HVAC) systems are used to provide heating or cooling, the problem therefore is one of designing good control schemes that can smoothly operate the HVAC systems in response to the needs of the occupants. This requires that (1) the heating/cooling needs of the occupants of the environmental space be quantified, (2) the dynamic characteristics of the given HVAC system be understood. Given this knowledge, the control system design must seek to balance the supply of heating and/or cooling from the HVAC system to the demand from the environmental space.

While there could be several admissible control schemes to achieve the above objective, we insist on choosing the one that is optimal. This additional requirement of optimality is desired because we can then redefine our objectives from comfort at any cost to comfort at minimum cost. This requires the application of optimal control theory to design what are known as optimal controllers. We explore the design of one such optimal controller in this paper.

One of the difficulties in the design of optimal controllers for space heating/cooling is the fact that even the simplest of the HVAC system models are nonlinear. To overcome this difficulty, good linearized models must be developed so that linear optimal control theory can be applied by assuming that the systems are operated on a small signal basis. Of course the advantage is that the optimal solution gives rise to optimal control-law of the type of linear state variable feedback. Furthermore, optimally designed linear systems can tolerate nonlinearities without impairment of their desirable properties [1].

To illustrate the design method, we consider a space heating problem. Note that the method is equally applicable to space cooling as well. To be more specific, we consider a simple system from the general class of HVAC systems; namely, a heat pump system (Fig. 1) with a storage tank used to supply heat to the environmental space. The heat pump receives source heat from ground water, (other sources such as ambient air or waste heat from other sources can be used as well), and elevates this source energy to a higher temperature, which is used to heat the water in the storage tank. The space heating is accomplished by circulating the hot water in the baseboard radiators.

The three principal subsystems that make up the complete space heating system, shown in Fig. 1, are: (1) the heat pump (primary plant), (2) the distribution system (circulating pump and baseboard radiators) and (3) the environmental space. In general, three different types of disturbances occur on the environmental space: (i) variations in the outdoor air temperature, (ii) temporal variation in the solar radiation fluxes enetering the space and (iii) the variations in the internal heat gains due to occupancy patterns of the space. All these three disturbances affect the space heating requirements. The question is: given the space heating system shown in Fig. 1, what is the optimal heat pump control input U_2^* which will take the system states (such as zone air temperature) from a given initial state to as close as possible to the desired state (such as zone set point temperature) without excessive expenditure of energy? In order to answer this question, the heating system problem must be formulated as a standard regulator problem. The solution to the finite time regulator problem will give an optimal law for controlling the input energy to the heat pump.

There have been some studies done on the use of heat pumps and their control in space heating applications. For example, Krause [2] studied the use of phase change heat storage with heat pumps for space heating. Wang *et al.* [3] designed a proportional-integral controller to regulate the capacity of the heat pump. Digital control of heat pumps for minimizing power consumption is examined by Parnitzki [4] and suboptimal control of a heat pump/heat storage system was examined in our earlier study [5]. On the other hand, the application of optimal control theory to environmental control problems is investigated by Rink *et al.* [6], Townsend *et al.* [7], Farris and McDonald [8] and Bloomfield and Fisk [9]. Townsend *et al.* [7] considered the optimal control of a general environmental space but their study did not consider the dynamics of the primary plant. Farris and McDonald [8] have used adaptive optimal control techniques to investigate the energy conservation potential in solar heated/cooled buildings. This approach is very

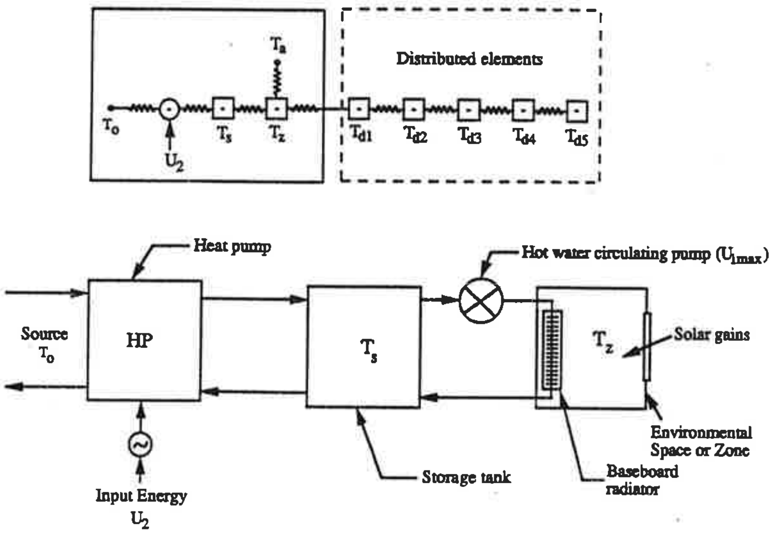

Fig. 1. Block diagram of the space heating system.

useful, except that it requires frequent model updating. Also, as shown by Zaheer-uddin [10], the use of estimated models obscure the physical understanding of the problem and is susceptible to errors when disturbances on the system change rapidly. Here, we present the application of optimal control theory to design an optimal controller for a single zone environmental space. In particular, we consider (1) higher-order models to simulate the space loads and (2) include the dynamics of the primary plant and its interactions with the environmental space. This type of approach will be suitable for extending this study to multi-zone systems for large buildings.

In the following, we formulate the space heating problem as a standard regulator problem. The results from the solution to the regulator problem will be given and the response of the optimal controller subject to impulse and step disturbance will be examined. Also, the design of a suboptimal controller will be presented and the response of the resulting closed-loop system with the reduced order estimator will be discussed.

2. ANALYTIC FORMULATION

Figure 1 shows the schematic diagram of a heat pump heat storage arrangement used in space heating. The heat pump draws energy from a low temperature energy source such as ground water or ambient air source and elevates this energy to a higher temperature, which is used to heat the water in the storage tank. A circulating pump (U_{1max}) is used to pump the hot water in the baseboard radiators for space heating. Also shown in Fig. 1 (inset) is a nodal representation of the system. By identifying the heat losses, heat gains and storage fluxes at each node, energy balance equations are written. If C_s is the thermal capacity of the storage tank and T_s its temperature, an energy balance on node T_s gives

$$C_s \frac{dT_s}{dt} = U_2 U_{2max} COP - U_{1max} a_{sb}(T_s - T_z) - a_s(T_s - T_z) \quad (1)$$

where the rate of heat stored in the storage tank is equated to the rate of heat delivered by the heat pump $U_2 U_{2max} COP$ to the storage tank, the rate of heat supplied to the space $U_{1max} a_{sb}(T_s - T_z)$ and the rate of heat loss from the tank to the space $a_s(T_s - T_z)$. The coefficient of performance (COP) of the heat pump is a function of the source temperature (T_o) and the storage tank temperature (T_s), and was modelled as

$$COP(T_s) = \begin{cases} 1 + (COP_{max} - 1)\left(1 - \dfrac{T_s - T_o}{\Delta T_{max}}\right), \\ 1, \quad \text{if} \quad T_s - T_o > \Delta T_{max}. \end{cases} \quad (2)$$

In equation (2), COP_{max} refers to the rated COP of the heat pump and ΔT_{max} refers to the maximum temperature differential the heat pump is designed to work with. At and beyond this limit the heat pump COP is taken to be equal to 1, which means when this limit is reached, the output from the heat pump is equal to the input energy.

Several methods are available for predicting space loads and hence T_z. Here, we use the methodology described in a previous study [11], in which the heat transfer processes occurring between the environmental zone and the outdoor environment were modelled. It was shown that these processes can be grouped into three major components: (i) the heat loss or gain through elements that could be assumed to be purely resistive, (ii) the heat loss or gain through lumped capacity elements (example, thin materials of the enclosure) and (iii) the heat transfer through thick materials such as concrete walls of the enclosure. This latter component must be modelled using a distributed capacity approach. In this paper, we neglect the effects of lumped capacity elements since they are small compared to those from distributed capacity elements.

If C_z is the zone interior air thermal capacity and T_z its temperature, an energy balance on node T_z gives

$$C_z \frac{dT_z}{dt} = U_{1max} a_{sb}(T_s - T_z) + a_s(T_s - T_z) - a_z(T_z - T_a) + I + hA_{d1}(T_{d1} - T_z), \quad (3)$$

where the rate of heat stored in the zone air is equated to the heat delivered from the storage tank $U_{1max} a_{sb}(T_s - T_z)$, the standby losses from the tank to the space $a_s(T_s - T_z)$, the heat losses from the space to outdoor ambient temperature $a_z(T_z - T_a)$, the internal heat sources I if any, and the storage fluxes added to or removed from the space $hA_d(T_{d1} - T_z)$. This latter component describes the typical thermal behaviour of thick mass walls, which is described by five nodes [11] (T_{d1} to T_{d5} in Fig. 1). The equations describing the heat transfer process occurring at each of the five nodes were written as

$$C_{d1} \frac{dT_{d1}}{dt} = -hA_{d1}(T_{d1} - T_z) - \frac{KA}{\Delta X}(T_{d1} - T_{d2}) + \alpha A_w S_g, \quad (4)$$

$$C_{d2} \frac{dT_{d2}}{dt} = -\frac{KA}{\Delta X}(T_{d2} - T_{d1}) - \frac{KA}{\Delta X}(T_{d2} - T_{d3}), \quad (5)$$

$$C_{d3} \frac{dT_{d3}}{dt} = -\frac{KA}{\Delta X}(T_{d3} - T_{d2}) - \frac{KA}{\Delta X}(T_{d3} - T_{d4}), \quad (6)$$

$$C_{d4} \frac{dT_{d4}}{dt} = -\frac{KA}{\Delta X}(T_{d4} - T_{d3}) - \frac{KA}{\Delta X}(T_{d4} - T_{d5}), \quad (7)$$

$$C_{d5} \frac{dT_{d5}}{dt} = -\frac{KA}{\Delta X}(T_{d5} - T_{d4}). \quad (8)$$

The disturbances on the system appear via the ambient temperature T_a and internal heat source I in equation (3) and solar gain term $\alpha A_w S_g$ term in equation (4).

2.1. *Formulation of the regulator problem*

Equations (1–8) describe a seventh-order bilinear model of the heat pump heating system. The model equations (1–8) were put in a standard regulator form, by linearizing them about an operating point of the system such as

$$\mathbf{T} = \mathbf{T}_0 + \Delta\mathbf{T} \quad (9)$$

where $\mathbf{T}$ represents the states T_z, T_s, T_{d1}, T_{d2}, T_{d3}, T_{d4}, and T_{d5} respectively. Similarly, the control input U_2 is also linearized so that

$$U_2 = U_{20}+\Delta U_2. \tag{10}$$

The linearized state equations were written in vector matrix form as

$$\Delta\dot{\mathbf{T}} = \mathbf{A}\Delta\mathbf{T}+\mathbf{B}\Delta U_2+\mathbf{E}\Delta\mathbf{W}, \tag{11}$$

and the output equation was defined as

$$y = \mathbf{H}\Delta\mathbf{T}, \tag{12}$$

where the matrices **A**, **B**, **H** and **E** are given by

$$\mathbf{A} = \begin{bmatrix} -C_{16} & C_{15} & C_{14} & & & & \\ C_9 & -C_{10} & 0 & & & & \\ C_{17} & 0 & -C_{19} & C_{18} & & & \\ & & C_{20} & -2C_{20} & C_{20} & & \\ & & & C_{20} & -2C_{20} & C_{20} & \\ & & & & C_{20} & -2C_{20} & C_{20} \\ & & & & & C_{20} & -C_{20} \end{bmatrix}, \tag{13}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ C_8-C_s T_o \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{14}$$

$\mathbf{H} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$ and the disturbance vector was written as

$$\mathbf{E}\Delta\mathbf{W} = \begin{bmatrix} C_{12} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta T_a + \begin{bmatrix} C_{21} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta I + \begin{bmatrix} 0 \\ 0 \\ C_{22} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta S_g. \tag{15}$$

The magnitude of the coefficients C_i were computed by using the system parameters defined in Table 1. Given the initial time t_0, the initial state $\Delta\mathbf{T}(t_0)$ and the final time T_f and the final state $\Delta\mathbf{T}(t_f) = 0$, the question is: what is the optimal control ΔU_2^* which will take the states $\Delta\mathbf{T}(t)$ from $\Delta\mathbf{T}(t_0)$ to a state as close to the origin as possible without excessive expenditure of control energy? This problem is equivalent to minimizing the following cost functional

$$J = \frac{1}{2}\int_{t_0}^{t_f} [\Delta\mathbf{T}'\mathbf{Q}\Delta\mathbf{T}+\Delta U_2' R \Delta U_2]\,\mathrm{d}t, \tag{16}$$

where the terminal cost is neglected.

Table 1. System parameters

	Variable	Magnitude and units
a_z	Building (zone) heat loss coefficient	740 kJ h^{-1} °C
a_s	Storage tank heat loss coefficient	8.8 kJ h^{-1} °C
C_s	Storage tank heat capacity	837 kJ °C^{-1}
C_z	Building air mass heat capacity	370 kJ °C^{-1}
a_{sb}	Heat transfer coefficient	0.6 kJ kg^{-1} °C
$U_{1\max}$	Pump capacity	3700 kg h^{-1}
$U_{2\max}$	Heat pump rated capacity	3.8 kW
$COP_{\max}$	Maximum heat pump COP	3.5
$\Delta T_{\max}$	Maximum temperature differential of heat pump	45°C
T_o	Source temperature	5°C

3. NUMERICAL SOLUTION TO THE REGULATOR PROBLEM

The solution to the regulator problem yields the optimal control law

$$\Delta U_2^* = -\mathbf{K}\Delta\mathbf{T}, \tag{17}$$

where the gain vector **K** is given by

$$\mathbf{K} = -R^{-1}\mathbf{B}'\mathbf{S}(t), \tag{18}$$

and $\mathbf{S}(t)$ is obtained by solving the matrix Riccati equation

$$\dot{\mathbf{S}}(t) = -\mathbf{S}\mathbf{A}-\mathbf{A}'\mathbf{S}+\mathbf{S}\mathbf{B}R^{-1}\mathbf{B}'\mathbf{S}-\mathbf{Q}, \tag{19}$$

with the boundary condition

$$\mathbf{S}(t_f) = 0. \tag{20}$$

Before attempting to solve (19), it is instructive to examine the controllability of the system. For this purpose, the matrices **A** and **B** were computed by using the system parameters given in Table 1. The structural features of the system described by **A** and **B** together with the numerical evaluation of the controllability of the pair (**A**, **B**) point to the fact that, the degree of control that can be exercised by the input U_2 on certain states of the system could be indeed marginal. For example, although there are seven time constants associated with the system, from a physical viewpoint they can be grouped into three: (1) the zone air time constant τ_z, (2) the storage tank time constant τ_s and (3) the time constant of the distributed elements τ_d. For the open-loop system, the approximate magnitudes were found to be $\tau_z = 120$ s, $\tau_s = 327$ s and $\tau_d = 4522$ s. This means, the overall system consists of (1) a fast subsystem and (2) a slow system. The control input U_2 has a direct effect on the fast subsystem states T_z and T_s but has an indirect effect (via natural convection coupling between zone air and mass wall) on the states T_{d1} through T_{d5}. A consequence of this is that the input to the system has a small and slow effect on the states T_{d1} through T_{d5}. Under these circumstances, at least three different alternatives can be explored. (1) design an optimal controller for the full-order system by overcoming, to some degree, the numerical problems, (2) design an optimal controller by developing a new model of the plant in which the effects of slow subsystem are incorporated into the fast subsystem and (3) explore a simple but suboptimal solution to this problem. While we have presented the optimal controller design based on approach (2) in another study [12], here we propose to explore the alternatives (1) and (3).

To this end, we propose the following strategy: (1) design an optimal controller for the system described by equations (11)–(16), (2) break the system states into two groups: (a) active states (T_z, T_s) (b) passive states (T_{d1} through T_{d5}); with the system states grouped into two, design an optimal controller based on only the active states of the system and implement it on the original system consisting of both active and passive states. Of course, this will lead to a suboptimal design. We present results based on optimal (seventh-order plant) and as well as suboptimal (reduced-order plant) designs so that meaningful comparisons can be made. With this as the motivation, the numerical solution to the Riccati equation (seventh-order model) was obtained and the optimal control law was implemented.

The results depicted in Fig. 2 correspond to the case in which the fluctuations in the zone air temperature were penalized more severely than those in the storage tank temperature. The rationale was that, in space heating applications, the objective is to control the space (zone) air temperature as close to the setpoint (origin) as possible. At the same time the magnitude of the weighting R on the control input ΔU_2 was kept large enough to avoid input saturation. In other words, the intent was to not exceed the capacity of the heat pump in an effort to maintain the zone air temperature at the set point.

Figure 2 shows the variations in the optimal state and control variables from the operating point. The system operating point was selected so as to correspond to an outdoor temperature of 20°C. Since the desired temperature of the zone was also taken to be 20°C, it is apparent that at this outdoor temperature the system input will be zero ($\Delta U_2 = 0$) implying no heating will be required. When an arbitrary initial disturbance is imposed on the system (such as when the zone temperature is decreased by 1°C—that is, from 20 to 19°C) the heat pump will supply heat to the space via the storage tank to restore the zone temperature back to 20°C. In Fig. 2, $\Delta T_z = \Delta T_s = \Delta T_{d1} = 0$ (for clarity the states ΔT_{d2} through ΔT_{d5} are not shown) represents the system operating point (20°C). Thus, $\Delta T_z = 0.5$ is equivalent to a zone air temperature of 20.5°C. The control input ΔU_2 in Fig. 2 represents the fraction of maximum input power supplied to the heat pump. A value of $\Delta U_2 = 0.2$ means the heat pump input power is equal to 20% of its rated input. The ΔU_2 sequence shown therefore represents as to how one might want to operate the heat pump when an impulse disturbance of 1°C is applied.

It may be noted that the time required to raise the zone temperature by 1°C to within ± 0.1°C of the setpoint temperature is about 275 s. This rise time is a function of (1) the thermal capacity of the storage medium, (2) the zone heat loss rate, (3) the storage capacity of the zone air volume, (4) the capacity (size) of the heat pump and (5) the magnitude of the weighting matrix **Q** and

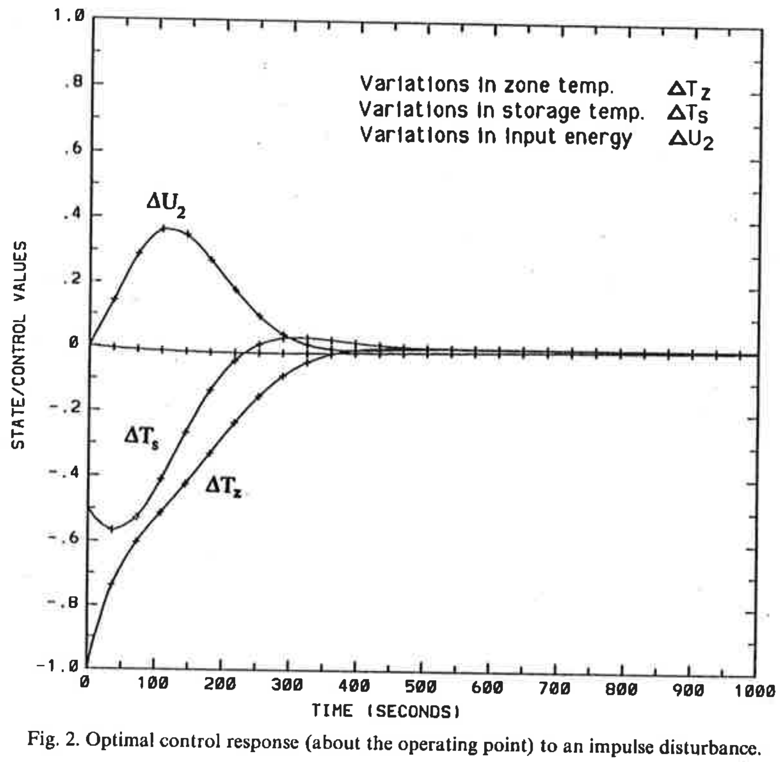

Fig. 2. Optimal control response (about the operating point) to an impulse disturbance.

scalar R. In fact for a given design, option (5) is the only alternative available to seek a balance between overshoot and steady state time. For example, it was found that an increase in $\mathbf{Q}$ by 200% resulted in a 22% decrease in steady state time and a 73% increase in the overshoot.

The impulse response tests are useful in assessing the space heating system response when switched on from a cold start. For example, many HVAC systems are turned-off during unoccupied periods, and turned-on ahead of the occupancy time so that the space temperature is brought to the setpoint when occupants arrive. The impulse response test then gives an indication of the lead time that could be necessary for an optimal start.

Although the system response to an impulse signal is good, in practice space heating systems under continuous operation are subjected to step-like disturbances. To study the effect of a unit step disturbance, the state feedback was implemented. It is apparent from Fig. 3 that the state feedback control gives rise to a steady state error in ΔT_z under a constant sustained disturbance. In order to remedy this situation, it is necessary to add an output error integrator to the system.

3.1. *Optimal regulator with integral feedback*

In order to eliminate the steady state error in T_z, an optimal controller with output integral feedback was designed. To this end, to the model equations given by (11)–(15), an error integrator

$$\dot{\xi} = 0\xi + e \tag{21}$$

was added. The error e was defined as

$$e = y - y_d. \tag{22}$$

With the addition of the integrator, the order of the system increases by one, that is we are now dealing with an eighth-order system. In vector matrix notation the model is described as

$$\begin{bmatrix} \Delta\dot{\mathbf{T}} \\ \Delta\dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{H} & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{T} \\ \Delta\xi \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \Delta U_2 + \begin{bmatrix} \mathbf{E} \\ 0 \end{bmatrix} \Delta\mathbf{W} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_d. \tag{23}$$

The quadratic cost functional is now defined as

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left\{ [\Delta\mathbf{T}'\Delta\xi'] \begin{bmatrix} \mathbf{Q}_1 & 0 \\ 0 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{T} \\ \Delta\xi \end{bmatrix} + \Delta U_2' R \Delta U_2 \right\} \mathrm{d}t. \tag{24}$$

The solution to the above regulator with integral control yields the control law

$$\Delta U_2^* = -[\mathbf{K}_1\ K_2] \begin{bmatrix} \Delta\mathbf{T} \\ \Delta\xi \end{bmatrix}, \tag{25}$$

where $\mathbf{K}_1$ contains the state feedback gains and K_2 the integral error gain.

It was found that the numerical solution to the optimal regulator with integral feedback is somewhat sensitive to

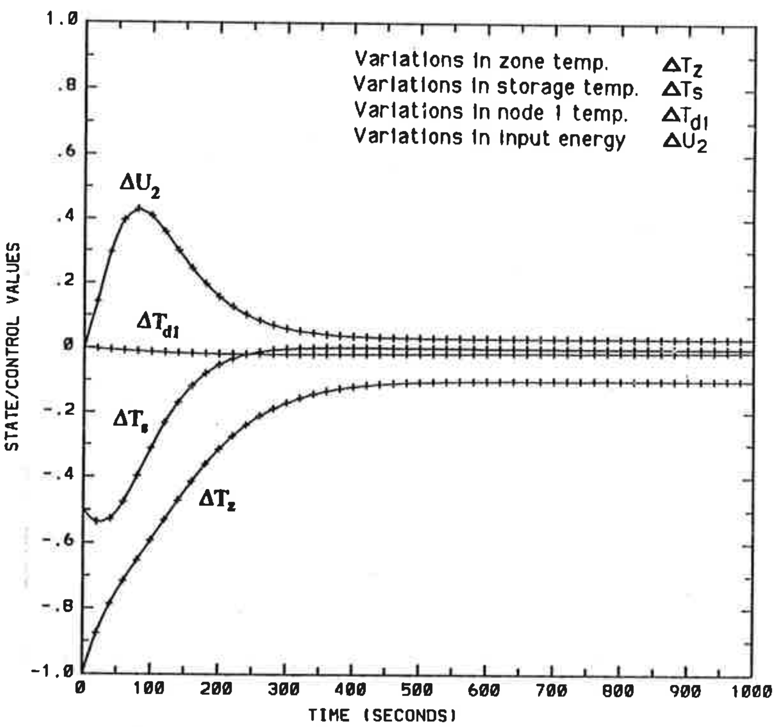

Fig. 3. The effect of a unit step disturbance (about the operating point) on the system.

the magnitude of the weighting factor, $\mathbf{Q}_2$. Test runs indicated that the weighting on the output integrator has to be kept fairly small to achieve faster settling time.

The response of the optimal closed-loop system to a unit step disturbance is shown in Fig. 4 (the infinite time regulator solution). It is apparent that the output integrator is able to eliminate the steady state error. The steady state condition is indicated by the constancy of ΔT_z and ΔT_s. The magnitude of $\Delta T_s > 0$ at steady state is due to the fact that the storage tank must provide a constant amount of energy to keep ΔT_z at the setpoint value under sustained disturbance. The input energy required for this purpose is also constant as shown by $\Delta U_2 > 0$ in Fig. 4.

Figure 4 also shows the relative effect of input ΔU_2 on the system states ΔT_z, ΔT_s and ΔT_{d1}. The states ΔT_{d2} through ΔT_{d5} are not shown because they did not change noticeably from their initial values (ΔT_{d2} to $\Delta T_{d5} \approx 0$). As pointed out before, the effect of ΔU_2 on ΔT_s and ΔT_z is significant, implying that they are easier to control. On the other hand, the effect of ΔU_2 on ΔT_{d1} (also ΔT_{d2} to Δt_{d5}) is very small indeed. This, in part, substantiates our earlier observation on the controllability of the plant. As such, we propose to use this particular characteristic of the system in order to simplify the design problem. As pointed out earlier, for the purpose of designing an optimal controller, a reduced order model (considering the dominant states ΔT_z, ΔT_s only) may be used and the state feedback can be implemented on the original eighth-order plant. Of course, this approach leads to a suboptimal design. Nevertheless, it simplifies the design problem considerably.

3.2. *Reduced-order plant*

For the purpose of suboptimal design, we consider only the dominant states T_z and T_s and construct a second-order plant model. When an output integrator is added to this system, the linearized reduced-order model is given by

$$\begin{bmatrix} \Delta \dot{T}_z \\ \Delta \dot{T}_s \\ \Delta \dot{\xi} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta T_z \\ \Delta T_s \\ \Delta \xi \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \end{bmatrix} \Delta U_2 + \begin{bmatrix} e_1 \\ 0 \\ 0 \end{bmatrix} \Delta \mathbf{W}, \quad (26)$$

where the plant output is ΔT_z.

The interest here is to compare the unit step responses obtained from the original plant (with integrator) (23) and the reduced-order plant (with integrator) (26). For simplicity, the steady-state gain vectors obtained from the solution to the respective infinite time regulator problems were used for feedback implementation. The results are shown in Figs 4 (full-order system) and 5 (reduced-order system). it may be noted that the general nature of the responses are similar; the difference is in the overshoot in the system states (ΔT_z, ΔT_s) and consequently in the magnitude of ΔU_2. In particular, the higher overshoot

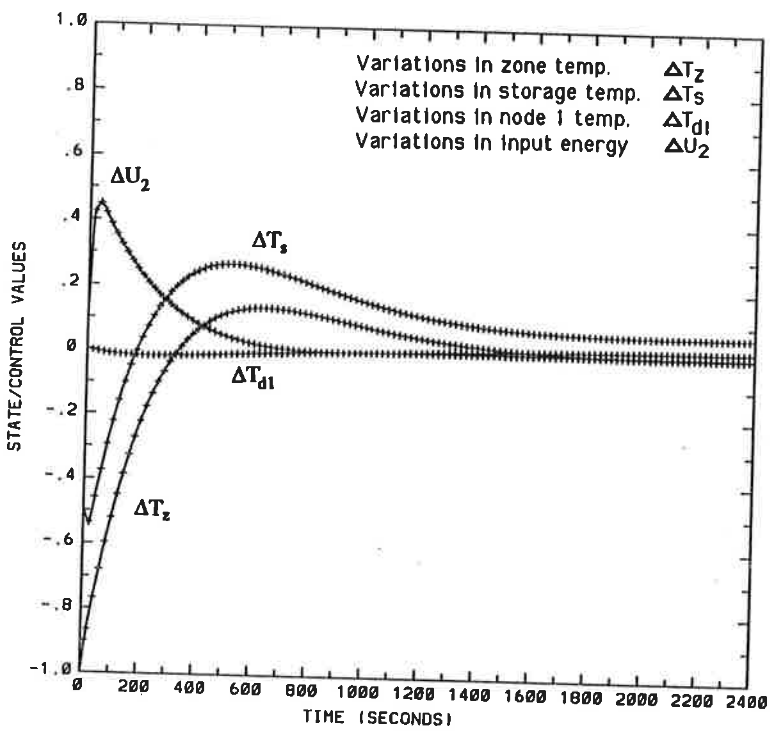

Fig. 4. The use of optimal output integral controller to eliminate the steady state error due to a step disturbance.

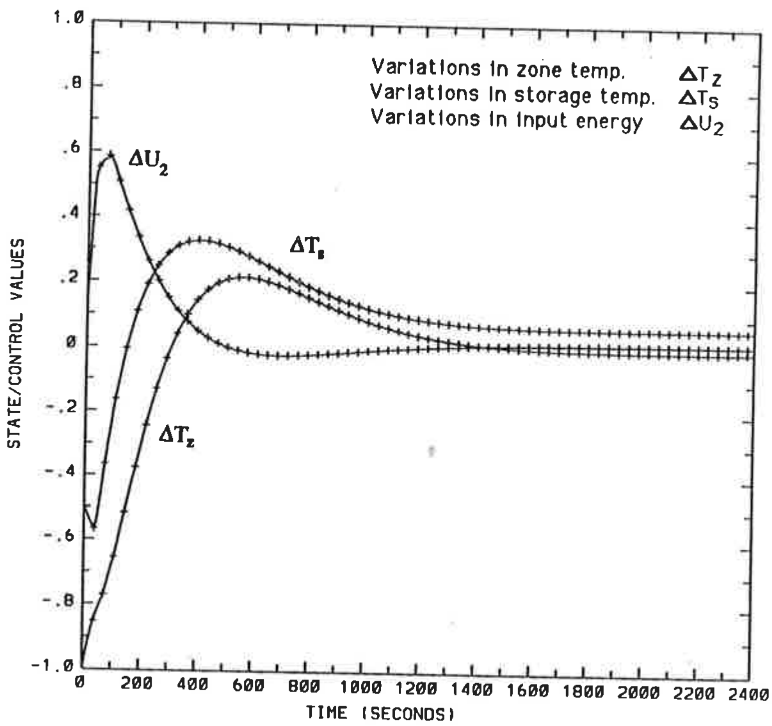

Fig. 5. Optimal control response (about the operating point) based on reduced-order model.

from the reduced-order system is due to the fact that the thermal capacity effects that help to store and retrieve energy in the full-order model are absent in the reduced-order model. As such, somewhat large variations in ΔT_s, ΔT_z are expected. The overshoot can be minimized while keeping $\Delta U_2 > 0$ by searching for new weighting matrix **Q** and the scalar R that gives an acceptable transient response. But the interest here is to implement this controller on the full order plant as discussed next.

While the reduced-order model does not describe the space heating problem as accurately as the original model, the similarity of the outputs shown in Figs 4 and 5 gives some indication that it can be used as a design tool. In order to illustrate this application, we implement the optimal feedback control law

$$\Delta U_2^* = -[K_1\ K_2\ K_3]\begin{bmatrix}\Delta T_z\\ \Delta T_s\\ \Delta\xi\end{bmatrix}, \tag{27}$$

obtained from the solution to reduced-order plant (26), on the original plant (23). The result of this implementation for a step disturbance in ΔT_a is shown in Fig. 6. The consequence of this is that, out of the seven states of the full-order plant plus integrator (23), only three states (two states plus integrator output) are used in the feedback. In other words, the remaining states (ΔT_{d1} through ΔT_{d5}) are assumed to be passive states; which means, they have no direct effect on ΔU_2 but do exercise an indirect control via ΔT_z and ΔT_s. The results depicted in Fig. 4 (state feedback based on all eight states) and Fig. 6 (state feedback based on three states) are in good agreement, except for some minor differences. Given this fact we emphasize that, not only does this approach simplify the design problem, but it also seems to be appropriate for indoor environment control problems. The reason is that, human thermal comfort is not influenced by small changes in zone air temperatures and therefore, very tight control on ΔT_z is not recommended. From this viewpoint, the approach and consequently the response obtained through its application (shown in Fig. 6) seems to be acceptable.

What happens, if in addition to the unit step disturbance in outdoor temperature, a unit step change occurs in the solar gains entering the space, is simulated in Fig. 7. It may be noted that ΔT_{d1} increases and reaches a steady state value in about 2000 s. Furthermore, even under the presence of three disturbances, simultaneously acting on the system (impulse, unit step in ΔT_a and unit step in ΔS_g), our objective of maintaining constant ΔT_z is not sacrificed. The space temperature is brought to within ± 0.1°C of the setpoint in 400 s and this error is completely eliminated in about 1600 s.

4. IMPLEMENTATION

In the preceding section the response of the optimal and suboptimal controllers was discussed. These results were based on the assumption that all states of the system are available for measurement. However, this is not always feasible nor is it economical to measure all the states. Particularly noteworthy is the heating system problem in which the temperatures of the distributed

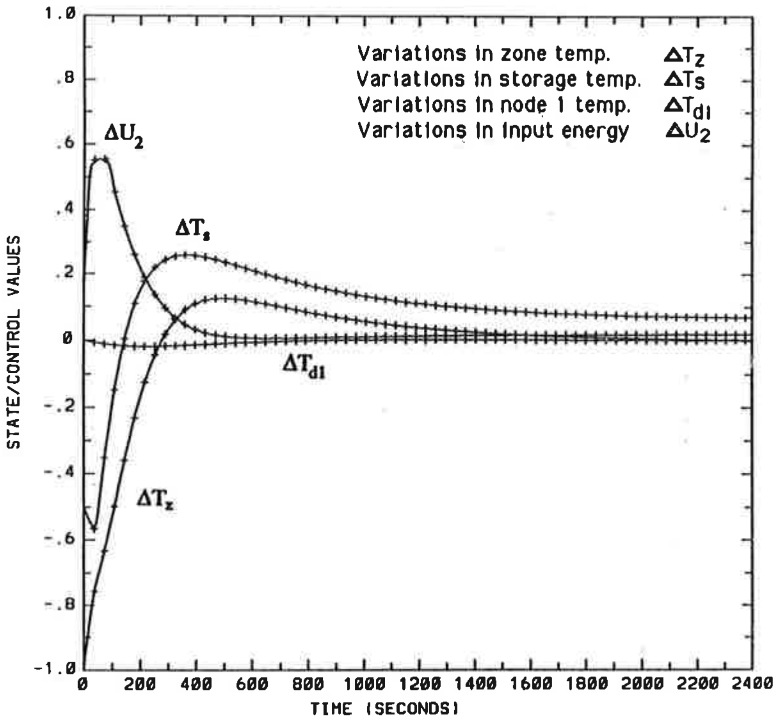

Fig. 6. The output response (about the operating point) of full-order plant driven by control-law designed from reduced-order model.

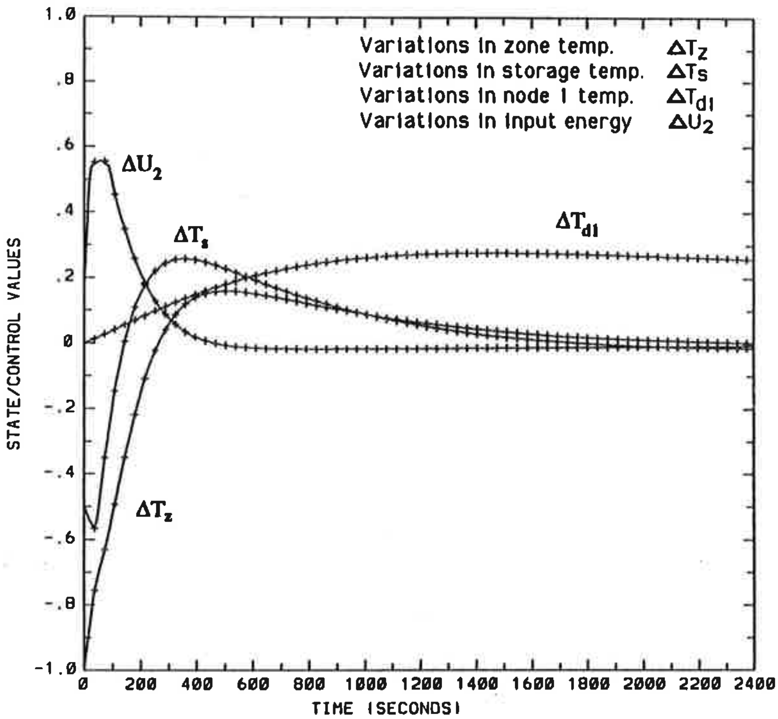

Fig. 7. Output response (about the operating point) to simultaneous step disturbances in ΔT_a and ΔS_g.

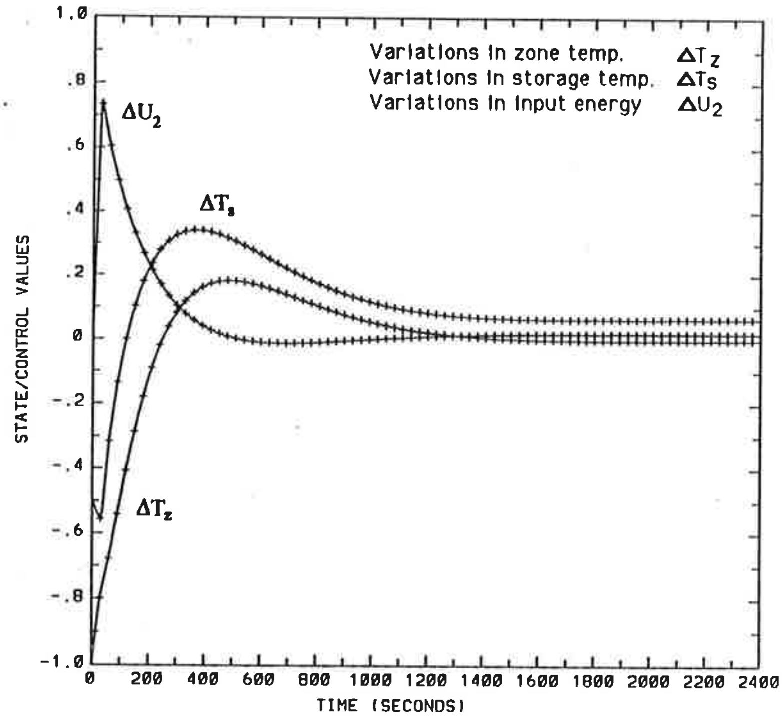

Fig. 8. The output response (about the operating point) of the plant with the observer.

thermal nodes (ΔT_{d1} to ΔT_{d5} in Fig. 1) are not accessible for measurement. Under these circumstances, it is necessary to design a reduced order estimator for the purpose of implementation.

Rather than dealing with the original system for the design of an observer, we recommend the same suboptimal approach discussed earlier. This means that, for the reduced-order model, which is completely observable, a first-order observer will be designed to estimate one of the states (ΔT_s). This observer will then be used to implement the suboptimal control on the original plant (23).

Therefore, the observer design problem can be stated as follows: given that the system output is ΔT_z and the state to be estimated is $\Delta \hat{T}_s$, design an observer of the form

$$\Delta \dot{Z} = f\Delta Z + g_1 \Delta T_z + g_2 \Delta U_2 \tag{28}$$

for the reduced-order plant (26). Once f, g_1 and g_2 are determined, the state feedback control

$$\Delta U_2 = -[K_1\ K_2\ K_3]\begin{bmatrix}\Delta T_z \\ \Delta \hat{T}_s \\ \Delta \xi\end{bmatrix} \tag{29}$$

can be implemented. In equation (29) **K** is the optimal gain vector obtained by solving the steady-state regulator problem for the plant described by (26) and the estimated state is given by

$$\Delta \hat{T}_s = \Delta T_z + L\Delta Z, \tag{30}$$

where L is the observer gain used to assign the pole of the observer at some desired location.

Figure 8 shows a good output response obtained by trial and error when the observer (28) and the feedback control (29) were implemented on the original plant (23). Ideally, this response should be close to the response shown in Fig. 6, where the state feedback on plant (23) was based on all three measured states. Of course some differences are expected due to the introduction of the observer dynamics. However, this difference, as shown in Figs 6 and 8 is limited to a few initial sample times.

To recapitulate, the response shown in Fig. 8 is from the original plant (23), whose state feedback is based on only three states out of a total eight. Moreover, of the three states used in the feedback, two were measured and the third one was estimated.

4.1. *Scope and limitations*

Although the design methodology presented in this paper can be extended to central heating, ventilating and air conditioning (HVAC) systems, several issues must be addressed before this can be accomplished. For example, good analytical models for HVAC systems have to be developed. The issues concerning the design of multiple input robust controllers subjected to multiple disturbances such as those occur in HVAC systems have to be addressed. As far as the regulator design presented in this study is concerned, it must be noted that it is valid for a single zone environmental space and is based on local linearization of the plant model about an operating point. This means that model updating is required whenever the system operating point changes. An alternative method for avoiding this is known as global linearization which utilises nonlinear feedback. The application of nonlinear feedback for single and multizone environmental spaces is discussed in another study [12].

5. CONCLUSIONS

The linear quadratic optimal regulator theory has been applied to the problem of optimizing the operation of a heat pump heating system when the energy storage capability exists in two forms: as active storage in the form of a hot water storage tank and passive storage as a result of the heat storage in the enclosure elements (walls, floor etc.) of the environmental space (zone). The response of the plant to impulse and step disturbance is shown to be good.

By invoking physical and controllability arguments, it is shown that, a near optimal controller for the full-order (original) plant can be designed from an approximately equivalent reduced-order plant. The suboptimal controller has been implemented on the original plant and the resulting response is shown to be almost as good as the response obtained from the same plant when optimal feedback control is applied.

It has further been shown that, the design of a reduced-order estimator for the original full-order plant can be based on the equivalent reduced-order plant. The response of the plant with the estimator is shown to be good.

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