Multi-Chamber Theory of Air Infiltration

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Estimates of air infiltration in houses based on tracer gas measurements have usually assumed the house to be a single perfect mixing chamber in which incoming air is instantaneously and uniformly diffused to all parts of the interior. In fact, some parts of the house—the basement or rooms with doors closed—exchange air only very slowly with other parts so that actual mixing is far from instantaneous. This paper presents the theory and mathematics necessary to apply the tracer gas method to buildings of many chambers.

TRACER gas measurements of air infiltration have usually assumed that the building is a single, perfect mixing chamber in which incoming air is instantaneously and uniformly diffused to all parts of the interior. In many cases this is a satisfactory assumption and much valuable air infiltration work has been based on it[1,2]. It often happens, however, that some parts of the building—the basement or rooms with doors closed—exchange air only very slowly with other parts, so that actual mixing is far from instantaneous. If different parts of the building aerate at different rates, then the tracer gas concentration, even if initially uniform, may become decidedly nonuniform as time passes. Under these circumstances, measurements taken at a single point may give misleading estimates of the true infiltration rate. Even with internal doors open, mixing may be somewhat obstructed by partitions. Hunt and Burch[3] for example, report differences between upstairs and downstairs tracer gas concentration decay patterns in a four bedroom house.

Buildings can often be more satisfactorily represented as a finite number n of inter-connected perfect mixing chambers, where $n > 1$. The minimum n that must be assumed will depend, among other things, on how the internal doors are set. If these are all open, then n may be as low as 2 (basement and rest of house), but if some are closed n may have to be higher.

This paper presents the mathematical groundwork for the n-chamber model in a tutorial fashion. In order to be as self-contained as possible, it includes brief proofs of some important results.† The first section formulates the problem in the way that is simplest from the mathematical point of view: it derives tracer gas concentrations from given inter-chamber flows. The second section treats the more difficult inverse problem which is actually encountered in infiltration studies: to derive the inter-chamber flows from observations of tracer gas concentrations. The third section discusses alternative experimental techniques which can greatly ease the analytic difficulties.

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†Although many of these proofs can be found, in some form, in standard works on mathematics they are often embedded in notation and contexts which make them somewhat inaccessible to the non-specialist. See, however, refs.[4] and [5] for the basic theory.

I. DERIVING TRACER GAS CONCENTRATIONS FROM GIVEN FLOWS

1. Diffusion among n chambers

Consider n chambers as indicated in Fig. 1 connected by one-way passages through which gas is flowing at given fixed rates, and suppose mixing within chambers is
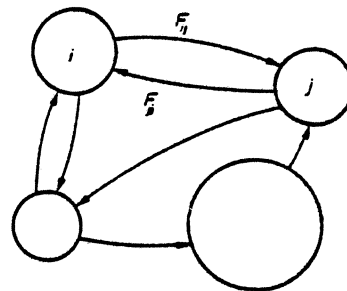

Fig. 1. Several interconnected chambers.

perfect and instantaneous. Suppose further that varying amounts of tracer gas have been injected into the different chambers. If the initial concentrations are $x_1(0), \ldots, x_n(0)$, what are the concentrations $x_1(t), \ldots, x_n(t)$ at later times t?

Let

V_i = volume of chamber i

F_{ij} = rate of flow from chamber i to chamber j. ($F_{ii} = 0$)

x_i = concentration of tracer gas in chamber i.

Then,

$F_{ij}x_i$ = rate of mass transfer of tracer from chamber i to chamber j

and

$V_i\dot{x}_i$ = rate of mass increase within chamber i.

Conservation of mass, then, gives:

$$V_j\dot{x}_j = \sum_i F_{ij}x_i - \left(\sum_k F_{jk}\right)x_j. \qquad (1)$$

mass transfer in   mass transfer out

Since total flow into chamber j must equal total flow out, we can write (using S_j to denote the common value):

$$\sum_i F_{ij} = \sum_k F_{jk} = S_j. \qquad (2)$$

flow in   flow out

The matrix of coefficients of the right-hand side of equation (1) becomes then:

$$F = \begin{pmatrix} -S_1 & F_{21} & F_{31} & \cdots & F_{n1} \\ F_{12} & -S_2 & F_{32} & \cdots & F_{n2} \\ F_{13} & F_{23} & -S_3 & \cdots & F_{n3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ F_{1n} & F_{2n} & F_{3n} & & -S_n \end{pmatrix}.$$

the matrix of the left-hand side is

$$V = \begin{pmatrix} V_1 & 0 & 0 & \cdots & 0 \\ 0 & V_2 & 0 & \cdots & 0 \\ 0 & 0 & V_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & V_n \end{pmatrix}$$

and equation (1), written in vector-matrix notation, becomes:

$$V\dot{x} = Fx. \tag{3}$$

This is a system of first order, linear differential equations. However, it is not just any system, since the matrices V and F have the following special properties:

(1) Matrix V is a diagonal matrix with non-negative elements.

(2) Matrix F has negative diagonal elements and non-negative off-diagonal elements.

(3) In matrix F, all row sums and all column sums are zero. This follows directly from flow conservation as expressed by equation (2).

From these properties we will derive a number of useful facts about the solution of equation (3), i.e. about the time evolution of the concentration in the n chambers:

$$x(t) = (x_1(t), x_2(t), \ldots, x_n(t)).$$

A basic property of the system of differential equations (3) is the following: for certain special initial concentrations $x(0)$ the solution has the simple form:

$$x(t) = x(0)e^{\lambda t} \tag{4}$$

that is: $x_1(t) = x_1(0)e^{\lambda t}, \ldots, x_n(t) = x_n(0)e^{\lambda t}$.

In this special case, the concentrations maintain fixed ratios to one another at all times. In general, there are n such special solutions (unique to within a scale factor), each with its own (or 'eigen-') value of λ. We will denote these special solutions as follows:

$$x^{(0)}e^{\lambda_0 t}$$
$$x^{(1)}e^{\lambda_1 t}$$
$$\vdots$$
$$x^{(n-1)}e^{\lambda_{n-1} t}.$$

Every other solution can be written as a linear com-

bination of these:

$$x(t) = \sum_{k=0}^{n-1} a_k x^{(k)} e^{\lambda_k t}. \tag{5}$$

To find the special initial conditions that lead to the simple form (4), we can substitute this form into the original equation (3). After differentiating and cancelling the common exponential factor, this yields:

$$\lambda Vx = Fx, \tag{6}$$

a static vector eigenvalue problem, whose eigenvectors are the required special initial conditions, $x^{(0)}, \ldots, x^{(n-1)}$.

The following facts about the eigenvalues and eigenvectors of (6) can be derived from the special properties of the matrices V and F listed above.

1. *One eigenvalue is always equal to zero and the corresponding eigenvector has equal components* (which can be normalized to unity):

$$\lambda_0 = 0, \quad x^{(0)} = (1, 1, \ldots, 1).$$

Proof. The left-hand side of (6) is zero because $\lambda = 0$. The right-hand side is zero because substitution of $x^{(0)}$ into the expression Fx yields the row sums of F, which are zero. Thus (6) is satisfied.

Significance. If all chambers initially have the same concentration, then nothing happens.
[Note that $\dot{x} = 0$, equation (3).] If arbitrary initial conditions $x(0)$ are expanded in eigenvectors [equation (5)]:

$$x(0) = a_0 x^{(0)} + a_1 x^{(1)} + \ldots,$$

then the coefficient a_0 is the mean concentration to which all $x_i(t)$ tend. Later we will let one of the chambers represent the outdoors ($V = \infty, x = 0$). Then the asymptotic concentration is always zero.

2. *The remaining eigenvalues are not necessarily real.* However, *complex eigenvalues and eigenvectors always occur in conjugate pairs*, i.e. if $\lambda + i\mu$ and $x + iy$ are an eigenvalue and corresponding eigenvector, then $\lambda - i\mu$ and $x - iy$ are also.

Proof. This follows simply from the fact that the matrices V and F are real. Substitution of complex unknowns, i.e. $x + iy$ and $\lambda + i\mu$, into (6) yields

$$V(\lambda x - \mu y) = Fx \quad \text{(real part)}, \tag{7}$$

$$V(\lambda y + \mu x) = Fy \quad \text{(imaginary part)}. \tag{8}$$

Substitution of the conjugate quantities $x - iy$, $\lambda - i\mu$ into (6) yields exactly the same equations.

Significance. Real initial conditions $x(0)$ are easily expressed as linear combinations of eigenvectors even though some eigenvectors are complex. If the coefficients of conjugate eigenvectors are real and equal, then the imaginary parts cancel out. Moreover, the vector $x(t)$ remains real as time passes. However, the time varying exponential factors associated with complex pairs contain

sinusoids and thus oscillate indefinitely.* More about oscillations below.

3. *All eigenvalues except $\lambda_0 = 0$ have negative real parts.* i.e. they lie in the left half of the complex plane.

Proof. Multiply equations (7) and (8) respectively by x and y and add. This yields

$$\lambda(xVx + yVy) = xFx + yFy, \qquad (9)$$

where the products have the meanings indicated below:

$$xVx = \sum_i V_i x_i^2$$

$$xFx = -\sum_j S_j x_j^2 + \sum_i \sum_j F_{ij} x_i x_j. \qquad (10)$$

Using equation (2), one can write

$$S_j = 1/2 \sum_i F_{ij} + 1/2 \sum_k F_{jk}.$$

Using this form in (10) and collecting terms in the right way, one obtains

$$xFx = \sum_i \sum_j (-1/2 F_{ij}(x_j^2 - 2x_i x_j + x_i^2))$$

$$= -\sum_i \sum_j 1/2(F_{ij} + F_{ji})(x_i - x_j)^2 < 0.$$

Similarly, $yFy < 0$. Thus the right hand side of (9) is negative, and since all factors on the left except λ are positive, λ must be negative. This shows that any non-zero eigenvalue has a negative real part, hence lies in the left half-plane.

Significance. This guarantees that all concentrations must tend asymptotically to a finite constant, that none can grow indefinitely. If any of the eigenvalues are complex, however, the decay may be accompanied by endless, though damped, oscillations as noted above. In the language of electrical engineering, the system may 'ring', albeit with ever diminishing amplitude. Since it may seem surprising that a diffusion process could ring, a very simple example with complex eigenvalues is given below. I hope this will make the possibility of ringing intuitively plausible.

2. Simple example with complex eigenvalues

$$n = 3$$

$$V_1 = V_2 = V_3 = 1$$

$$F_{12} = F_{23} = F_{31} = 1, F_{21} = F_{32} = F_{13} = 0$$

(see Fig. 2).
Matrices:

$$V = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$
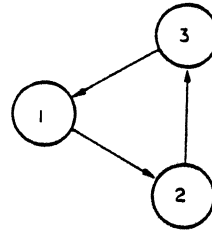

Fig. 2. Example with complex eigenvalues. Tracer gas concentrations $x_1(t), x_2(t), x_3(t)$ oscillate as they approach equality.

The eigenvector equation:

$$\lambda I x = F x \qquad \text{or} \qquad (\lambda I - F)x = 0$$

has a nonzero solution only if the determinant vanishes:

$$\begin{vmatrix} (\lambda+1) & 0 & -1 \\ -1 & (\lambda+1) & 0 \\ 0 & -1 & (\lambda+1) \end{vmatrix} = 0,$$

or:

$$(\lambda+1)^3 = 1.$$

The eigenvalues are the three solutions of this equation. If the cube roots of one are denoted by $1, E_1, E_2$, then the eigenvalues are:†

$$\lambda_0 = 1 - 1 = 0$$

$$\lambda_1 = E_1 - 1 = -3/2 + i\frac{\sqrt{3}}{2}$$

$$\lambda_2 = E_2 - 1 = -3/2 - i\frac{\sqrt{3}}{2}$$

The eigenvectors are also conveniently expressed in terms of the cube roots of one:

Eigenvalues	Corresponding eigenvectors
$\lambda_0 = 0$	$x^0 = (1, 1, 1)$
$\lambda_1 = E_1 - 1$	$x^1 = (1, E_2, E_1)$
$\lambda_2 = E_2 - 1$	$x^2 = (1, E_1, E_2)$

Any initial conditions can be expressed as a linear combination of eigenvectors:

$$x(0) = a_0 x^0 + a_1 x^1 + a_2 x^2.$$

The subsequent evolution of x then is

$$x(t) = a_0 x^0 + a_1 x^1 e^{\lambda_1 t} + a_2 x^2 e^{\lambda_2 t}.$$

For example, if the initial concentrations in the three chambers are

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*Euler's equation: $e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos \mu t + i \sin \mu t)$.

†The cube roots of 1, namely 1, $-(1/2) + i\sqrt{3/2}$, $-(1/2)$ $- i\sqrt{3/2}$ are spaced 120° apart on the unit circle in the complex plane.

$$x_1 = 0$$
$$x_2 = 3$$
$$x_3 = 3,$$

then it can be verified that the concentrations evolve as described below. Let points P_1, P_2, P_3 be evenly spaced on a circle as shown in Fig. 3. Imagine that the circle turns with constant angular velocity and simultaneously shrinks exponentially in radius. Then the points P_1, P_2, P_3 trace out logarithmic spirals as indicated. Their projections onto the x-axis are the concentrations, x_1, x_2, x_3 in the three chambers. It is evident that these concentrations oscillate indefinitely as they approach the equilibrium concentration $x = 2$. Intuitively, one can imagine a blob of concentrated gas cycling around and around through the three chambers as it simultaneously evens out becoming less and less pronounced. This illustrates, then, how even a diffusion process can 'ring'.
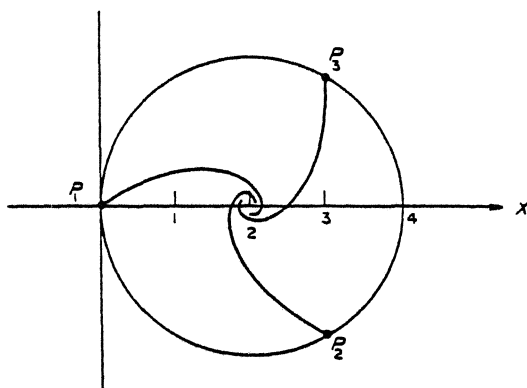

Fig. 3. The points P_1, P_2, P_3 move simultaneously inward on logarithmic spirals. Their projections on the x-axis represent the tracer gas concentrations in the three chambers.

3. Two chambers connected to the outdoors

In this section we calculate the eigenvalues and eigenvectors for the important three-chamber case in which one chamber has infinite volume and zero concentration of tracer gas, and thus serves to model the outdoors. The other two chambers may represent, for example, the basement and upstairs of a house (see Fig. 4).

The eigenvalues in this case are always real: the infinite volume chamber effectively blocks the kind of cycling that led to oscillatory solutions in the previous example.
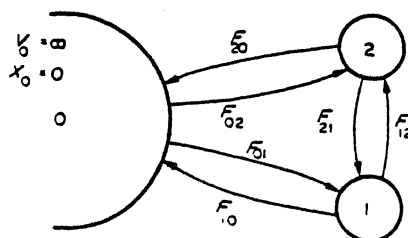

Fig. 4. The three chamber case where one chamber (the outdoors) has infinite volume and zero tracer gas concentration.

For $n = 3$, equation (3) becomes

$$V_0 \dot{x}_0 = -S_0 x_0 + F_{10} x_1 + F_{20} x_2$$
$$V_1 \dot{x}_1 = F_{01} x_0 - S_1 x_1 + F_{21} x_2$$
$$V_2 \dot{x}_2 = F_{02} x_0 + F_{12} x_1 - S_2 x_2.$$

The infinite volume $V_0 = \infty$ implies, by the first equation, that $\dot{x}_0 = 0$. Since the initial value of x_0 is zero, $x_0(t)$ remains zero for all subsequent time. The first equation and the first column of the right hand side may therefore be deleted. The eigenvalue equations of the remaining system are

$$-(S_1 + \lambda V_1) x_1 + F_{21} x_2 = 0$$
$$F_{12} x_1 - (S_2 + \lambda V_2) x_2 = 0.$$

Setting the determinant equal to zero yields the eigenvalue equation:

$$(S_1 + \lambda V_1)(S_2 + \lambda V_2) - F_{12} F_{21} = 0,$$

whose roots are the eigenvalues:

$$\lambda = -\frac{1}{2}\left(\frac{S_1}{V_1} + \frac{S_2}{V_2}\right) \pm \frac{1}{2}\left[\left(\frac{S_1}{V_1} + \frac{S_2}{V_2}\right)^2 - 4\left(\frac{S_1}{V_1}\frac{S_2}{V_2} - \frac{F_{12}}{V_2}\frac{F_{21}}{V_1}\right)\right]^{1/2}.$$

It is easy to verify by inspection that the eigenvalues are real and negative. The components x_1, x_2 of the eigenvector corresponding to an eigenvalue λ have the ratio:

$$\frac{x_2}{x_1} = \frac{S_1 + \lambda V_1}{F_{21}} = \frac{F_{12}}{S_2 + \lambda V_2}.$$

4. Numerical example

A numerical example is shown in Fig. 5. The relative sizes of the flows in this hypothetical case correspond roughly with observations of Twin Rivers houses. Figure 6 shows the two eigenvector solutions, which, as noted above, are the building blocks for all other solutions. Figure 7 shows two such other solutions: the top graph shows what happens when tracer gas is injected only into Chamber 1 (basement) and the bottom graph shows what happens when the tracer is injected only into Chamber 2 (living space). Note that after a long time has elapsed, the concentrations in *both* cases become nearly proportional to the components of the dominant eigenvector (the one whose eigenvalue is largest). This always happens, regardless of initial conditions, except in singular cases.

The next section discusses the inverse problem actually encountered in practice: how to deduce the values of the flows F_{ij} from the observed concentrations $x_j(t)$, i.e. how to deduce a picture such as Fig. 5.

II. INVERSE PROBLEM—DETERMINING FLOWS FROM OBSERVATIONS OF TRACER CONCENTRATIONS

1. Counting equations and unknowns

In a system of n chambers communicating with each other and with the outside, there are $(n+1)n$ flows F_{ij}.

(Regarding the outside as a chamber of infinite volume, there are $n+1$ chambers in all, and each receives a flow from the n others.) The $(n+1)n$ flows are, however, not independent because they must satisfy the n conservation equations

$$\sum_{i=0}^{n} F_{ij} = \sum_{i=0}^{n} F_{ji}, \quad j=1,\ldots,n, \ (F_{jj}=0).$$

Note that there are not $n+1$ of these because the conservation equation for the $(n+1)$st chamber (the outside):

$$\sum_{i=0}^{n} F_{i0} = \sum_{i=0}^{n} F_{0i},$$

is already implied by the first n equations.

To determine the F_{ij}, then, we need exactly n^2 further equations. We can obtain these from the differential equations (1), which can be rewritten in the form:

$$\sum_{i=1}^{n} F_{ij}(x_i - x_j) - F_{0j}x_j = V_j\dot{x}_j, \quad j=1,\ldots,n. \quad (11)$$

A single observation of the concentrations $x_1,\ldots,x_n$ and their derivatives $\dot{x}_1,\ldots,\dot{x}_n$ yields n equations for the F_{ij}. To obtain the necessary n^2 equations, the x_i and $\dot{x}_i$ must be observed at n different time points, i.e. the data

$$x_i(t_k), \dot{x}_i(t_k), i=1,\ldots,n; \quad k=1,\ldots,n \quad (12)$$

must be taken.

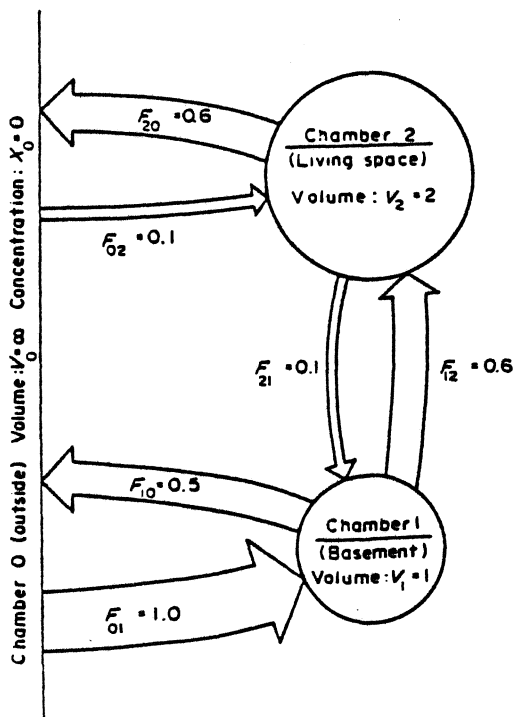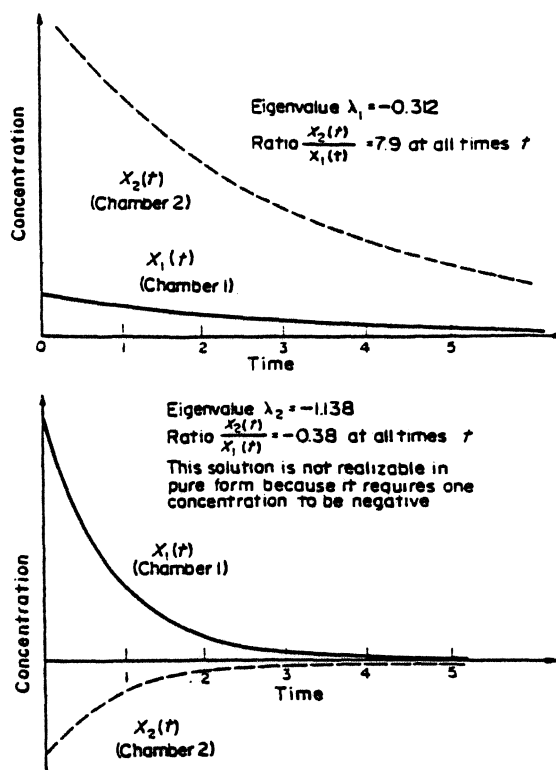Fig. 6. Eigenvector solutions for the example of Fig. 5. When the initial conditions coincide with an eigenvector (as in the cases shown), the concentrations maintain fixed proportions to one another and decay at the exponential rate λ.

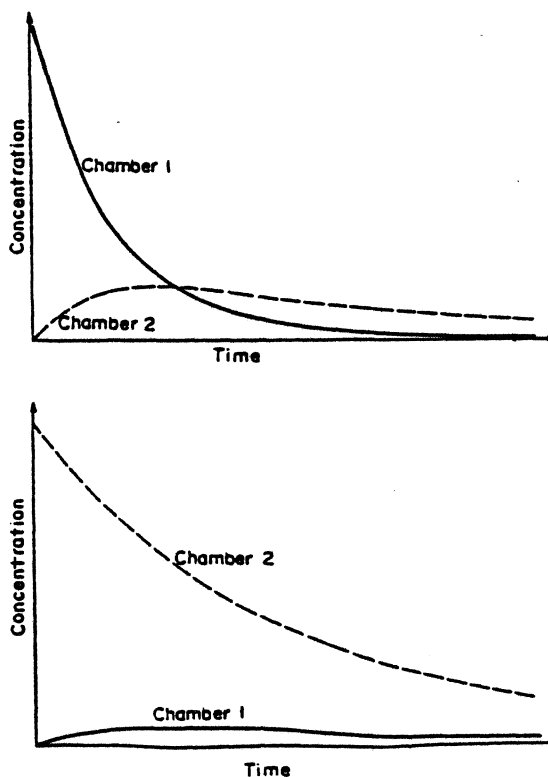Fig. 5. Numerical example. Units are arbitrary.

Fig. 7. Two other sets of initial conditions for the example of Fig. 5. These and all other solutions are linear combinations of the eigenvector solutions shown in Fig. 6.

In the case of a single chamber (usually assumed in past infiltration measurements), $n^2 = 1$ and a single observation of the pair $x, \dot{x}$ suffices to estimate the infiltration rate. As n increases, the number of necessary observations rapidly becomes unwieldy and several practical difficulties may occur:

(a) The observations of the derivatives $\dot{x}_i$ are likely to be unreliable because of their sensitivity to noise (applies also to the case $n = 1$).

(b) If the times $t_1, \ldots, t_n$ are close together, the system of n^2 equations will be badly determined (i.e. the determinant will be close to zero). If the times $t_1, \ldots, t_n$ are widely spaced, then the assumption that the flows F_{ij} remain fixed over the period from t_1 to t_n may be doubtful.

(c) If the initial conditions happen to coincide with an eigenvector, then the equation sets generated at different times are all the same, so that no more than n independent equations can be obtained. In general, the initial conditions must contain strong components of *all* eigenvectors or else the equations will be badly determined. It is important to recognize that the vector of concentrations $x_1, \ldots, x_n$ converges to the first eigenvector over time for almost any initial conditions. Thus not more than one of the time points t_k should be so far from the injection time that the concentrations are close to the first eigenvector. This constraint can be relieved somewhat by making two runs with different initial conditions (see Fig. 7), but this, of course, leans heavily on the assumption that the F_{ij} are time invariant.

(d) In general, all of the F_{ij} must be calculated in order to calculate any of them. But usually some of the F_{ij} are of more interest than the others. It would be good to have simple means for calculating only the ones of greatest interest.

2. Dealing with the difficulties

The noise can be suppressed somewhat by integrating equation (11) over time:

$$\sum_{i=1}^{n} F_{ij} \int_{T_1}^{T_2} (x_i(t) - x_j(t)) dt$$

$$- F_{0j} \int_{T_1}^{T_2} x_j(t) dt = V_j(x_j(T_2) - x_j(T_1)).$$

Integrating over n different time intervals gives n^2 equations for the F_{ij}. Since there are many different ways of choosing the n time intervals, there are many different systems of n^2 equations that can be set up (all in theory yielding the same values of F_{ij}). The intervals should be chosen to give the best compromise among three conflicting desiderata: (1) noise should be suppressed (the longer the intervals, the better), (2) the system should be well-determined (vectors $x_1, \ldots, x_n$ in different intervals linearly independent), (3) the total time interval covered should not be so long that drifts in the values of F_{ij} become important. This problem needs further examination to see if clever methods of estimation can be devised which exploit the special characteristics of air infiltration data.

With regard to the fourth difficulty listed above—the necessity of calculating all F_{ij} at once—it should be noted

that there is a special circumstance under which an important subset of the F_{ij} can be obtained simply and directly without obtaining the others. If there is a time t at which all of the concentrations have the same value x, then the n direct infiltration flows from the outside are given by

$$F_{0j} = V_j \frac{(-\dot{x}_j)}{x}, j = 1, \ldots, n.$$

This follows immediately from equation (11). The overall infiltration rate, of course, is just

$$\sum_j F_{0j}.$$

3. Solution of the inverse problem for two chambers connected to the outdoors

This is the case discussed in Part I and illustrated in Figs. 4–7. Since it often occurs in practice, its solution is of interest in its own right. It will also serve to illustrate the points made above.

The inverse problem is to calculate values for the six flows illustrated in Fig. 4 from the observed evolution of concentrations in the two chambers. If the tracer gas was initially injected into just one of the chambers, then the observed concentrations might look, for example, like those shown in Fig. 7.

The first step is to eliminate two of the flow variables by means of the conservation equations. There are many ways to do this, but a very simple way is just to drop the two outward flows F_{10} and F_{20}. It is not difficult to verify that the remaining four flows, $F_{01}, F_{02}, F_{12}, F_{21}$, are independent in the sense that they can be assigned any positive values without violating the condition that total flow into any chamber must equal total flow out. Once these four flows are determined, the two that were dropped can be calculated from the conservation equations:

$$F_{10} = F_{01} + F_{21} - F_{12},$$

$$F_{20} = F_{02} + F_{12} - F_{21}.$$

The four unknown flows can be expressed in terms of the eigenvalues and eigenvectors. For compactness we will represent the eigenvectors simply by the ratios of their two components:

$$r_1 = \frac{x_1^1}{x_2^1}.$$

$$r_2 = \frac{x_1^2}{x_2^2}.$$

These are ratios of Chamber 1 concentrations to Chamber 2 concentrations. They are the two special concentration ratios which, when once established, stay fixed over time. Since eigenvectors are only determined up to a multiplicative constant, the ratios r_1 and r_2 specify the eigenvectors completely. To express the four unknown flows in terms of the four quantities $\lambda_1, \lambda_2, r_1, r_2$, we can invert the equations given at the end of Section 3 of Part I. This inversion yields:

r lows from outdoors:

$$F_{01} = \frac{\left(1 - \frac{1}{r_1}\right)\lambda_2 - \left(1 - \frac{1}{r_2}\right)\lambda_1}{\frac{1}{r_1} - \frac{1}{r_2}} \, V_1$$

$$F_{02} = \left[\frac{(1 - r_1)\lambda_2 - (1 - r_2)\lambda_1}{r_1 - r_2}\right] V_2.$$

Flows between chambers:

$$F_{21} = \left[\frac{\lambda_1 - \lambda_2}{\frac{1}{r_1} - \frac{1}{r_2}}\right] V_1$$

$$F_{12} = \left[\frac{\lambda_1 - \lambda_2}{r_1 - r_2}\right] V_2.$$

It remains to estimate $\lambda_1, \lambda_2, r_1, r_2$ from the observed data. This is easy in principle, but in practice it may be difficult for the reasons given above. Let $x_1(t), x_2(t)$ be the observed time evolutions of the concentrations in the two chambers. Then, as can be verified using equation (5) the functions

$$y_1(t) = \ln\left[x_1(t) - r_1 x_2(t)\right]$$

$$y_2(t) = \ln\left[x_1(t) - r_2 x_2(t)\right]$$

are straight lines with slopes λ_2 and λ_1 respectively. To obtain estimates of r_1 and r_2 one can vary the parameter a in the expression

$$y(t, a) = \ln\left[x_1(t) + a x_2(t)\right]$$

and pick out those two values of a for which $y(t, a)$ is most linear. One of these values should be positive and the other negative, since $r_1 > 0$ and $r_2 < 0$. The slopes of the best fitting straight lines can be taken as estimates of λ_1 and λ_2. In carrying out this procedure one can be as simple or elaborate as one chooses. A simple method is to plot $x_1(t) - a x_2(t)$ on log paper using various values of a until a sufficiently linear plot is obtained. An elaborate method is to systematically fit $y(a, t)$ to straight lines by linear regression for a range of a's and pick those values of a which minimize the sum of the squared residuals.

A preliminary estimate of r_1 and λ_1 can be obtained simply by looking at the asymptotic behavior of $x_1(t)$ and $x_2(t)$. After a long time has passed, the ratio $x_2(t)/x_1(t)$ will be nearly constant and nearly equal to r_1 and both $x_1(t)$ and $x_2(t)$ will decay at nearly the exponential rate λ_1. On log paper, therefore, the plots of $x_1(t)$ and $x_2(t)$ will be asymptotic to straight lines of slope λ_1 whose constant separation is the ratio r_1. This is true regardless of initial conditions. It occurs because the second eigenfunction always dies out faster than the first. Unfortunately there is no correspondingly simple way of estimating r_2 and λ_2.

and, as can be seen from the equations given above for the four unknown flows, all of the eigenvectors and eigenvalues must be known in order to calculate any of the flows.

III. ALTERNATIVE EXPERIMENTAL METHODS WHICH REDUCE THE ANALYTICAL DIFFICULTIES

The experimental method discussed so far can be called the *transient method* because it examines the transient decay of tracer gas concentrations $x_1(t), \ldots, x_n(t)$ following a single initial injection. Alternatively one can consider *steady-state* methods in which the concentrations $x_1, \ldots, x_n$ are held fixed over time.* The quantities observed are the rates $r_1, \ldots, r_n$ at which tracer gas must be released in the n chambers in order to maintain the pre-specified concentrations. This is analogous to measuring the thermal leakiness of a house by observing the fuel required to maintain a fixed internal temperature.

Of special interest is the case in which all concentrations are held fixed and equal: $x_1 = \ldots = x_n = x$, because in this case the most important infiltration data can be read off directly with no calculation at all. If all concentrations equal x, then the n release rates $r_1, \ldots, r_n$ are proportional to the direct infiltration rates $F_{01}, \ldots, F_{0n}$ into the n chambers from outside. To see that this is the case, note that all air entering chamber i from other chambers is at the common concentration x, while air entering from outside is at zero concentration. Tracer gas is released in chamber i at just the rate needed to bring the outside air up to concentration x.† With suitable calibration the output numbers $r_1, \ldots, r_n$ can be made to read directly in ft³/min.

In order to apply this method, it is necessary to have a self-regulating machine which constantly monitors the concentration x and adjusts the release rate r accordingly. A difficulty is that the machine's response time cannot be faster than the mixing time within the chamber. The observed release rate r may, therefore, show spurious fluctuations related only to the servo mechanism and not to real fluctuations of the infiltration rate. Over periods that are long compared to mixing time, however, these fluctuations will tend to average out, so that long term average infiltration rates should be relatively accurate.

The n direct infiltration rates $F_{01}, \ldots, F_{0n}$ do not, of course, tell the whole story. However, the remaining F_{ij} can be determined by a modification of the procedure described above. To obtain the n outflows from chamber k, i.e. $F_{k0}, F_{k1}, \ldots, F_{kn}$, one places steady state machines in all chambers except chamber k and in chamber k one places a concentration-measuring machine. To see what this does, it is convenient to have the general steady-state equations for arbitrary but fixed concentrations x_i:

$$F_{0j}x_j + \sum_{i=1}^{n} F_{ij}(x_j - x_i) = r_j \quad j = 1, \ldots, n.$$

In each chamber j for $j \neq k$ the concentration x_j is set equal to 1 and the rate of gas release r_j is measured. In chamber k, $r_k = 0$ and x_k is measured. Putting these values into the equations yields

$$F_{0j} + F_{kj}(1 - x_k) = r_j \qquad j \neq k,$$

$$F_{0k}x_k - \sum_{i=1}^{n} F_{ik}(1 - x_k) = 0 \qquad j = k.$$

Assuming that the inflows F_{0j} have been determined by the previous method, F_{kj} are given, then, by:

$$F_{kj} = \frac{1}{1 - x_k}(r_j - F_{0j}) \qquad j \neq 0$$

$$F_{k0} = \frac{1}{1 - x_k}F_{0k} - \sum_{j=1}^{n} F_{kj}.$$

(Derivation of the last equation requires use of the flow conservation equation for chamber k.)

In the absence of a concentration-measuring machine, the *relative* sizes of $F_{kj}, j = 1, \ldots, n$ can still be determined: only the scale factor $1/(1 - x_k)$ will be unknown.

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